MATA31

Calculus I for Mathematical Sciences

Lecture Notes and Pertinent Proofs

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documented by Eric Wu

UNIVERSITY OF TORONTO

Instructor(s): **Dr. Michael Cavers**
Email: michael.cavers@utoronto.ca
Office: IA 4110

Contents

1 Lec02

1.1 Is a sum of increasing function also increasing?

I. W.T.S:

if f, g are increasing functions on an interval I, then $f+g$ is also increasing on an interval I

II. Assume f, g are increasing functions on an interval I, then

$$\begin{cases} \forall x_1, x_2 \in I, x_1 < x_2 \implies f(x_1) < f(x_2) \\ \forall x_1, x_2 \in I, x_1 < x_2 \implies g(x_1) < g(x_2) \end{cases}$$

by definition of increasing functions

III. let $h = f + g$
IV. W.T.S:

$$\forall x_1, x_2 \in I, x_1 < x_2 \implies h(x_1) < h(x_2)$$

V. Lemma (1) :

$$a < b \wedge c < d \implies a + c < b + d$$

V.a proof of Lemma (1)
Suppose $a < b \wedge c < d$, by inequality property (e) on review slide 18,

$$a + c < b + c \wedge b + c < b + d$$

by the transitive law,

$$a + c < b + d$$

VI. let x_1, x_2 be arbitrary
VII. suppose

$$x_1 < x_2$$

VIII. then,

$$h(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2) = h(x_2)$$

by Lemma (1)

IX. This means that

$$\forall x_1, x_2 \in I, x_1 < x_2 \implies h(x_1) < h(x_2) \text{ as required}$$

and if f, g are increasing functions on an interval I, then $f+g$ is also increasing on an interval I
X. The sum of increasing function is indeed increasing ■

1.2 Proof By Contradiction Practice

I. Show that the statement is true:

$$\forall x, y \in \mathbb{R}, x \in \mathbb{Q} \land y \notin \mathbb{Q} \implies x + y \notin \mathbb{Q}$$

II. Let

$$x, y \in \mathbb{R}$$

III. Suppose

$$x \in \mathbb{Q} \land y \notin \mathbb{Q}$$

IV. Suppose, also, ad absurdum

$$x + y \in \mathbb{Q}$$

V. Then, by definition of $\mathbb{Q}$,

$$x = \frac{p_1}{p_2} \land x + y = \frac{q_1}{q_2}$$

for some $p_1, p_2, q_1, q_2 \in \mathbb{Z} \land p_2, q_2 \neq 0$

VI.

$$y = -x + (x + y)$$

$$= -\frac{p_1}{p_2} + \frac{q_1}{q_2}$$

$$= \frac{q_1 p_2 - p_1 q_2}{p_2 q_2}$$

VI. by theorem $m, n \in \mathbb{Z} \implies mn \in \mathbb{Z}$

$$p_2 q_2, q_1 p_2, p_1, q_2 \in \mathbb{Z}$$

VII.by theorem $m, n \in \mathbb{Z} \implies m + n \in \mathbb{Z}$

$$q_1 p_2 - p_1 q_2 \in \mathbb{Z}$$

since $q_1 p_2 - p_1 q_2 = -p_1 q_2 + q_1 p_2$

VIII. by theorem $\forall a, b \in \mathbb{R}, ab \neq 0 \iff a \neq 0 \land b \neq 0$

$$p_2 q_2 \neq 0$$

XI. Thus, we have

$$y = \frac{p}{q}$$

where $p = q_1 p_2 - p_1 q_2 \in \mathbb{Z}$ and $q = p_2 q_2 \neq 0 \in \mathbb{Z}$

X.

$$y \in \mathbb{Q}$$

by definition of $\mathbb{Q} : \{\frac{m}{n} s.t. m, n \in \mathbb{Z}, n \neq 0\}$

XI. This contradicts supposition **III.**
we have shown that it is not the case that $x + y \in \mathbb{Q}$ and so $x + y \notin \mathbb{Q}$ as required.

■

2 Lec 3a

Absolute values and inequalities

Def^n 2.1 *Let* $x \in \mathbb{R}$
Then,

$$|x| = \begin{cases} x, \ if \ x \geq 0, \\ -x, \ if \ x < 0. \end{cases}$$

Or, let $x, y \in \mathbb{R}$

$$The \ distance \ between \ x \wedge y \ is \ |y - x|$$

Theorem 2.1.1 *Let* $x, y \in \mathbb{R}$
Then,

$$|x + y| \leq |x| + |y|$$

this is called **triangle inequality**.

2.1 Proof of Triangle Inequality

Proof.
Assume (i) - (v) have been proven in slide 2
Let $x, y \in \mathbb{R}$
We consider two cases.
Case 1.
Assume $x + y \geq 0$.
Then,

$$|x+y| = +(x+y)$$

by definition of $|\Box|$

$$\leq |x| + |y|$$

by (iv.) twice, i.e., $\begin{cases} x \leq |x| \\ y \leq |y| \end{cases}$

Case 2.
Assume $x + y < 0$.
Then,

$$|x+y| = -(x+y)$$

by definition of $|\Box|$

$$= -(x) - (y)$$

by algebra

$$\leq |-x| + |-y|$$

by (iv.) twice, i.e., $\begin{cases} -x \leq |-x| \\ -y \leq |-y| \end{cases}$

$$= |x| + |y|$$

by (iii.) twice, i.e., $\begin{cases} |-x| = |x| \\ |-y| = |y| \end{cases}$

3 Lec 3b

Limits

Def^n 3.1 *If we can make the values of f(x) arbitrarily close to L by taking x to be sufficiently close to c (on either side of c) but not equal to c, then*

$$\lim_{x \to c} f(x) = L$$

.

- If a limit exists, it is a single number.

- If a limit does not exist (D.N.E.), it could be that the limit $\begin{cases} \infty \\ -\infty \\ \text{one-sided limits differ} \\ \text{oscillates, e.g., } \frac{1}{\sin x} \end{cases}$

More formally:

Def^n 3.2 *Let $L, c \in \mathbb{R}$ and suppose that f is defined on an open interval around c (except possibly at c). Then,*

$$\lim_{x \to c} f(x) = L$$

which means $\exists L \in \mathbb{R}$ such that

if,

$$\forall \epsilon > 0, \exists \delta \text{ s.t. } 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$

3.1 An Example: Prove $\lim_{x\to 2}(3x-8) = -2$

I.WTS

$$\forall \epsilon > 0, \exists \delta > 0 \ s.t. \ 0 < |x-c| < \delta \implies |f(x) - L| < \epsilon$$

where, $f(x) = 3x - 8; L = -2; c = 2.$

Which means,

$$\forall \epsilon > 0, \exists \delta > 0 \ s.t. \ 0 < |x-2| < \delta \implies |3x-6| < \epsilon.$$

II. Let ϵ be arbitrary such that it is greater than zero.

III. Choose

$$\delta = \frac{\epsilon}{3}$$

note that $\delta > 0$ since $\epsilon > 0$.

IV. Assume

$$0 < |x-c| < \delta.$$

V. We have

$$0 < |x-2| < \delta = \frac{\epsilon}{3}$$

by chosen δ and given c.

VI. Then,

$$0 < 3|x-2| < 3\delta = \epsilon$$

multiplying inequality **V.** by 3

$$\implies |3||x-2| < 3\delta = \epsilon$$

by definition of $|\square|$

$$\implies |3(x-2)| < 3\delta = \epsilon$$

by property $|xy| = |x||y|$

$$\implies |3x-6| < 3\delta = \epsilon$$

by algebra

VII. Recall that

$$f(x) = 3x - 8; L = -2; c = 2.$$

VIII. We have

$$\forall \epsilon > 0, \exists \delta > 0 \ s.t. \ 0 < |x-2| < \delta = \frac{\epsilon}{3} \implies |3x-6| < \epsilon$$

as required.

■

3.2 Uniqueness Theorem

Theorem 3.2.1 *Let $L_1, L_2, c \in \mathbb{R}$ and suppose f is defined on an open interval around c (except possibly at c).*

$$\lim_{x\to c} f(x) = L_1 \land \lim_{x\to c} f(x) = L_2 \implies L_1 = L_2$$

Proof.
I. Assume

$$\lim_{x\to c} f(x) = L_1 \land \lim_{x\to c} f(x) = L_2.$$

II. Suppose, ad absurdum,

$$L_1 \neq L_2.$$

III. By definition of limit,

$$\begin{cases} \forall \epsilon_1 > 0, \exists \delta_1 > 0 \ s.t.\ 0 < |x-c| < \delta_1 \implies |f(x) - L_1| < \epsilon_1 \\ \forall \epsilon_2 > 0, \exists \delta_2 > 0 \ s.t.\ 0 < |x-c| < \delta_2 \implies |f(x) - L_2| < \epsilon_2. \end{cases}$$

III. Let

$$\epsilon = \frac{|L_1 - L_2|}{2}.$$

Note that $\epsilon > 0$ since $L_1 \neq L_1$ per assumption **II.**

IV. Let, also

$$\epsilon_1 = \epsilon_2 = \epsilon > 0.$$

IV. Then there exists

$$\delta_1, \delta_2 > 0 \text{ such that } |f(x) - L_1| < \epsilon_1 \land |f(x) - L_2| < \epsilon_2.$$

V. Choose

$$\delta = \min\{\delta_1, \delta_2\}.$$

Note that both $\delta \leq \delta_1 \land \delta \leq \delta_2$ hold.

VI. Choose $x \in \mathbb{R}$ such that $0 < |x - c| < \delta$.
Then we have both

$$0 < |x-c| < \delta_1 \land 0 < |x-c| < \delta_2$$

,and thus we have also

$$|f(x) - L_1| < \epsilon_1 = \epsilon \land |f(x) - L_2| < \epsilon_2 = \epsilon.$$

VII. Now, we sum $|f(x) - L_1| < \epsilon_1 = \epsilon \land |f(x) - L_2| < \epsilon_2 = \epsilon$ and get

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < 2\epsilon = |L_1 - L_2|$$

by triangle inequality and property $|f(x) - L_1| = |L_1 - f(x)|$.

VIII. Since **VII.**, $|L_1 - L_2| < |L_1 - L_2|$ is a contradiction we must refute supposition **II**.
Thus, indeed $L_1 = L_2$.

■

4 Lec 4a

Practice of limit proof:

Prove $\lim_{x\to 2}(3x^2 - 3x + 6) = 4$

I. WTS

$$\forall\epsilon > 0, \exists\delta > 0 \text{ s.t } 0 < |x-2| < \delta \implies |(3x^2 - 3x + 6) - 4| < \epsilon.$$

By definition of limit.

II. Let

$$\epsilon > 0 \text{ be arbitrary}$$

III. Choose

$$\delta = \min\{1, \frac{\epsilon}{2}\}.$$

Note that $\delta \leq 1 \wedge \delta \leq \frac{\epsilon}{2}$; also $\delta > 0$ since $\epsilon \wedge 1 > 0$

IV. Assume

$$0 < |x-2| < \delta.$$

V. Lemma (1)

$$|x-1| < 2$$

V.a Proof of Lemma (1).

V.b From $|x-1|$,

$$\begin{aligned} |x-1| &= |x-2+1| && \text{by algebra} \\ &\leq |x-2| + 1 && \text{by triangle inequality} \\ &< \delta + 1 && \text{since } |x-2| < \delta \\ &\leq 1 + 1 && \text{since } \delta \leq 1 \text{ by the chosen } \delta. \\ &= 2 \end{aligned}$$

□

VI. Thus,

$$\begin{aligned} |(3x^2 - 3x + 6) - 4| &= |(x-1)(x-2)| && \text{by algebra} \\ &= |x-1||x-2| && \text{by property } |xy| = |x||y| \\ &< 2|x-2| && \text{by Lemma (1)} \\ &< 2\delta && \text{by assumption } \textbf{IV.} \\ &\leq 2\frac{\epsilon}{2} && \text{by chosen } \delta \\ &= \epsilon && \text{by algebra} \end{aligned}$$

as required. ■

5 Lec 4b

Theorem

Let $L, M, a, b, c \in \mathbb{R}$, and suppose f and g are defined on an open interval around c (except possibly at c). If

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M,$$

then

$$\lim_{x\to c} af(x) + bg(x) = aL + bM.$$

Proof.
I. Assume

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M.$$

II. WTS

$$\forall \epsilon > 0, \exists \delta \text{ s.t. } 0 < |x-c| < \delta \implies |af(x) + b(fx) - (aL + bM)| < \epsilon.$$

III. By assumption **I.** and limit definition, we have

$$\begin{cases} \forall \epsilon_1 > 0, \exists \delta_1 \text{ s.t. } 0 < |x-c| < \delta_1 \implies |f(x) - c| < \epsilon_1 \\ \forall \epsilon_2 > 0, \exists \delta_2 \text{ s.t. } 0 < |x-c| < \delta_2 \implies |g(x) - c| < \epsilon_2. \end{cases}$$

IV. Let

$$\epsilon > 0 \text{ be arbitrary.}$$

V. Let

$$\epsilon_1 = \frac{\epsilon}{2(|a|+1)}, \epsilon_2 = \frac{\epsilon}{2(|b|+1)}.$$

VI. Choose $\delta_1, \delta_2 > 0$ satisfying

$$\delta = \min\{\delta_1, \delta_2\}.$$

VII. Assume

$$0 < |x-c| < \delta.$$

VIII. As $\delta \le \delta_1 \wedge \delta \le \delta_2$ it follows that $0 < |x-c| < \delta_1 \wedge 0 < |x-c| < \delta_2$, which suffices for the holding of

$$|f(x) - c| < \epsilon_1 \wedge |g(x) - c| < \epsilon_2,$$

by assumption **I.**

XI. Then,

$$\begin{aligned}
|af(x) + bg(x) - (aL + bM| &= |a(f(x) - L) + b(g(x) - M)| && \text{by algebra} \\
&\le |a||f(x) - L| + |b||g(x) - M| && \text{by triangle inequality and } |xy| = |x||y| \\
&< |a|\epsilon_1 + |b|\epsilon_2 && \text{by } \textbf{VIII.} \\
&= |a|\frac{\epsilon}{2(|a|+1)} + |b|\frac{\epsilon}{2(|b|+1)} && \text{by chosen } \epsilon_1, \epsilon_2 \\
&= \frac{|a|}{|a|+1} \cdot \frac{\epsilon}{2} + \frac{|b|}{|b|+1} \cdot \frac{\epsilon}{2} && \text{by algebra} \\
&< 1 \cdot \frac{\epsilon}{2} + 1 \cdot \frac{\epsilon}{2} && \text{by inequality property } a, b, c > 0 \implies \tfrac{a}{b+c} < \tfrac{a}{c} \\
&= \epsilon && \text{by algebra; as per required.}
\end{aligned}$$

■

6 Lec 5a

6.1 Squeeze Theorem

Let $L, c \in \mathbb{R}$ and there's $d > 0$ such that functions f, g, h are defined on a punctured interval around c.

Theorem 6.0.1 *If*

$$\begin{cases}(1) g(x) \leq f(x) \leq h(x), \forall x \in (c-d, c) \cup (c, c+d) \\ (2) \lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L\end{cases}$$

then,

$$\lim_{x \to c} f(x) = L$$

<u>Proof.</u>

I. Assume

$$(1)\ g(x) \leq f(x) \leq h(x), \forall x \in (c-d, c) \cup (c, c+d)$$
$$(2) \lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

hold.

II. WTS

$$\forall \epsilon, \exists \delta \text{ s.t. } 0 < |x-c| < \delta \implies |f(x) - L| < \epsilon$$

by definition of limit.

III. Let

$$\epsilon > 0 \text{ be arbitrary.}$$

IV. By assumption (2) and definition of limit we have

$$\begin{cases}\forall \epsilon_1, \exists \delta_1 \text{ s.t. } 0 < |x-c| < \delta_1 \implies |g(x) - L| < \epsilon_1 \\ \forall \epsilon_2, \exists \delta_2 \text{ s.t. } 0 < |x-c| < \delta_2 \implies |h(x) - L| < \epsilon_2.\end{cases}$$

V. Let

$$\epsilon_1 = \epsilon, \epsilon_2 = \epsilon.$$

VI. Choose

$$\delta = \min\{\delta_1, \delta_2, d\}.$$

VII. Assume

$$0 < |x - c| < \delta.$$

VIII. It follows that

$$0 < |x-c| < \delta_1 \wedge 0 < |x-c| < \delta_2 \wedge 0 < |x-c| < d$$

which suffice for

$$|g(x) - L| < \epsilon_1 \wedge |h(x) - L| < \epsilon_2 \wedge g(x) \leq f(x) \leq h(x).$$

IX. By our chosen ϵ_1, ϵ_2, this means

$$\begin{aligned}
& |g(x) - L| < \epsilon \wedge |h(x) - L| < \epsilon \wedge g(x) \leq f(x) \leq h(x) \\
\implies & L - \epsilon < g(x) < L + \epsilon \wedge L - \epsilon < h(x) < L + \epsilon \wedge g(x) \leq f(x) \leq h(x) && \text{by definition of } |\cdot| \\
\implies & L - \epsilon < g(x) \leq f(x) \leq h(x) < L + \epsilon && \text{by algebra} \\
\implies & L - \epsilon < f(x) < L + \epsilon && \text{by algebra} \\
\implies & |f(x) - L| < \epsilon && \text{by definition of } |\cdot|\text{; as required.}
\end{aligned}$$

■

7 Lec 5b

7.1 Continuity

Def$^{\underline{n}}$: f is continuous at c
if
$$\lim_{x\to c} f(x) = f(c)$$
i.e.,
$$\begin{cases} (1)\ f(c) \text{ is defined} \\ (2)\ \lim_{x\to c} f(x) \text{ exists; i.e., } \lim_{x\to c^-} f(x) = \lim_{x\to c^+} \\ (3)\ \lim_{x\to c} f(x) = f(c). \end{cases}$$

Def$^{\underline{n}}$: f is continuous at c
if
$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

7.2 Practice of proving continuity

Prove $f(x) = x^2$ is continuous on $\mathbb{R}$.
Proof.
I. Let

$$c \in \mathbb{R} \text{ be arbitrary.}$$

II. WTS

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } |x - c| < \delta \implies |x^2 - c^2| < \epsilon$$

by definition of continuity.

III. Let

$$\epsilon > 0 \text{ be arbitrary.}$$

IV. Choose δ satisfying

$$\delta = \min\{1, \frac{\varepsilon}{(1 + 2|c|)}\}$$

note that $\delta > 0$.

V.Assume

$$|x - c| < \delta.$$

VI. We have

$$\begin{aligned} |x + c| &= |(x - c) + 2c| && \text{by algebra} \\ &\leq |x - c| + 2|c| && \text{by triangle inequality} \\ &< \delta + 2|c| && \text{by assumption } \mathbf{V.} \\ &\leq 1 + 2|c| && \text{by our chosen } \delta. \end{aligned}$$

VII. Then

$$\begin{aligned} |x^2 - c^2| &= |(x + c)(x - c)| && \text{by algebra} \\ &= |x + c| \cdot |x - c| && \text{by absolute value property} \\ &< (1 + 2|c|) \cdot \delta && \text{by assumption } \mathbf{VI.} \text{ and algebra in } \mathbf{VII.} \\ &\leq (1 + 2|c|) \cdot \frac{\varepsilon}{(1 + 2|c|)} && \text{by chosen } \delta \\ &= \varepsilon && \text{by algebra; as required} \end{aligned}$$

■

7.3 Theorems

$$\lim_{x\to 0} \cos x = 1 \tag{1}$$

$$\forall x \in \mathbb{R}, |\sin x| \leq |x| \tag{2}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{3}$$

Proof of (3)
<u>Proof.</u>
I. Apply Squeeze Theorem.
II. For the lower bound,

II.a <u>case 1</u> $\forall x \in [0, \frac{\pi}{2}]$,

$$
\begin{aligned}
\frac{1}{2}x(1)^2 &\leq \frac{\tan x}{2} && \text{by algebra}\\
\implies x &\leq \tan x && \text{by algebra}\\
\implies x &\leq \frac{\sin x}{\cos x} && \text{by algebra}\\
\implies \cos x &\leq \frac{\sin x}{x} && \text{by algebra}
\end{aligned}
$$

II.b <u>case 2</u> $\forall x \in [-\frac{\pi}{2}, 0]$,

$$
\begin{aligned}
\frac{1}{2}x(1)^2 &\leq \frac{\tan x}{2} && \text{by algebra}\\
\implies x &\leq \tan x && \text{by algebra}\\
\implies x &\leq \frac{\sin x}{\cos x} && \text{by algebra}\\
\implies x\cos x &\geq \sin x && \text{as } \cos x < 0\\
\implies \cos x &\leq \frac{\sin x}{x} && \text{as } x < 0
\end{aligned}
$$

II.c Thus,

$$\forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}], \cos x \leq \frac{\sin x}{x}$$

as required.

8 Lec 7a

8.1 Archimedean Property

Theorem 8.0.1 *For every $x, y \in \mathbb{R}$ with $x, y > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.*

Proof.

- Let $x, y \in \mathbb{R}$ be arbitrary.
- Assume $x, y > 0$.
- To derive a contradiction, assume $\forall n \in \mathbb{N}, nx \leq y$. (i.e., "assume the opposite").
- Consider $S = \{nx \mid n \in \mathbb{N}\}$.
- Then y is an upper bound of S (by definition of upper bound).
- Since $S \neq \emptyset$ and is bounded above, S has a least upper bound (by the LUB Property).
- Let $\alpha = \sup(S)$.
- Then $\alpha - x$ is not an upper bound of S since $\alpha - x < \alpha$ (recall $x > 0$).
- In particular, there must be an element of S (call it nx) bigger than $\alpha - x$, that is, $\alpha - x < nx$ for some $n \in \mathbb{N}$.
- Thus, $\alpha < nx + x = (n+1)x$.
- Since $n + 1 \in \mathbb{N}$, we have that $(n+1)x \in S$.
- Thus, α is not an upper bound of S since $\alpha < (n+1)x$ and $(n+1)x \in S$.
- This is a contradiction since α is the least upper bound.
- Therefore, it must be that $\exists n \in \mathbb{N}$ such that $nx > y$. ■

8.2 Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 8.0.2 *For every two real numbers a, b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

Proof.

- Let $a, b \in \mathbb{R}$ be arbitrary.
- Assume $0 < a < b$.
- By the Archimedean Property (use $x = b - a > 0$ and $y = 1 > 0$), there exists $n \in \mathbb{N}$ such that $n(b - a) > 1$.
- Choose m to be the smallest integer greater than na.
- That is, find $m \in \mathbb{N}$ such that $m - 1 \leq na < m$.
- Then $na < m < nb$ (since $nb - na > 1$).
- Thus, $na < m < nb$ implying that $a < \frac{m}{n} < b$.
- Choose $r = \frac{m}{n} \in \mathbb{Q}$ which has the required property.
- To complete the proof, also consider the cases $a = 0$ and $a < 0$. ■

8.3 Boundedness Theorem

Theorem 8.0.3 *If f is continuous on $[a, b]$, then f is bounded on $[a, b]$.*

Proof.

- Assume f is continuous on $[a, b]$.
- Consider the set $S = \{x \in [a, b] : f \text{ is bounded on } [a, x]\}$.
- Note that S is non-empty (since $a \in S$) and bounded above by b.
- By the completeness property of the real numbers, S has a least upper bound, say $c = \sup(S)$. We claim that $c = b$.
- To derive a contradiction, suppose that $c < b$.
- Note that $c > a$ since f is right-continuous at a (and so there is a $\delta_1 > 0$ such that f is bounded on $[a, a + \delta_1]$).
- Since f is continuous at c, there exists a $\delta > 0$ such that f is bounded on $[c - \delta, c + \delta]$.
- Thus, f is bounded on $[a, c - \delta]$ and on $[c - \delta, c + \delta]$, implying f is bounded on $[a, c + \delta]$.
- This contradicts the choice of c as the least upper bound of S. Hence, $c = b$.
- This means that f is bounded on $[a, x]$ for all $x < b$.
- By the right-continuity of f at b, there exists a $\delta > 0$ such that f is bounded on $[b - \delta, b]$.
- Thus, f is bounded on $[a, b - \delta]$ and $[b - \delta, b]$, which implies that f is bounded on $[a, b]$. ■

8.4 Extreme Value Theorem

Theorem 8.0.4 *If $f(x)$ is continuous on the closed interval $[a, b]$, then $f(x)$ must attain both an absolute maximum and an absolute minimum on $[a, b]$.*

Proof.

- Assume f is continuous on $[a, b]$.
- By the Boundedness Theorem, f is bounded on $[a, b]$ (i.e., $\{f(x) \mid x \in [a, b]\}$ is bounded).
- Let $M = \sup(\{f(x) \mid x \in [a, b]\})$. (This exists by the Least Upper Bound Property.)
- (Want to show: M is the maximum, i.e., there exists a $c \in [a, b]$ such that $f(c) = M$.)
- To derive a contradiction, suppose there is no such c.
- Then $f(x) < M$ for all $x \in [a, b]$.
- Define a new function $g(x) = \frac{1}{M - f(x)}$.
- Since $M - f(x) > 0$ for all $x \in [a, b]$, we have $g(x) > 0$.
- Additionally, $g(x)$ is continuous on $[a, b]$ since $f(x)$ is continuous on $[a, b]$.
- By the Boundedness Theorem, g is also bounded on $[a, b]$.
- Therefore, there exists some $K > 0$ such that $-K \leq g(x) \leq K$ for every $x \in [a, b]$.
- Consequently, $\frac{1}{M - f(x)} \leq K$, which implies that $M - f(x) \geq \frac{1}{K}$ for all $x \in [a, b]$.
- Thus, $f(x) \leq M - \frac{1}{K}$ for all $x \in [a, b]$.
- This, however, contradicts the assumption that M is the least upper bound of $f(x)$ on $[a, b]$.
- Therefore, there must exist some $c \in [a, b]$ such that $f(c) = M$.
- Hence, f attains its maximum on $[a, b]$.
- The proof that f attains its minimum on $[a, b]$ follows similarly. ■

9 Lec 7b

9.1 Intermediate Value Theorem:

If f is a continuous function on the closed interval $[a, b]$ and N is any number strictly between $f(a)$ and $f(b)$, then there exists a number c in (a, b) s.t. $f(c) = N$

Implication/ Application

1. Every polynomial with an odd degree has at least one real root (i.e., solution)
2. To approximate a solution to $f(x) = 0$ (assuming one exists):

• Find an interval $[a, b]$ on which the function changes sign.
• Evaluate f at the midpoint $\frac{a+b}{2}$ and choose whichever subinterval f changes sign on.
• Repeat to get smaller and smaller subintervals.

Proof. (When$N = 0$)

I. Assume f is a continuous function on the closed interval $[a, b]$ and $f(a) < 0 < f(b)$.
II. Consider the set $S = \{\gamma \in [a, b] : f \text{ is negative on} [a, \gamma)\}$
III. Recall the definition of continuity: if f is continuous at c:

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

Note that, this means if f is right continuous,
then

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } x - c < \delta \implies |f(x) - f(c)| < \epsilon.$$

Also, for left continuity

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } -\delta < x - c < \implies |f(x) - f(c)| < \epsilon.$$

IV. Note that f is right continuous at a. We take

$$\epsilon = -f(x) > 0$$

note that $\epsilon > 0$.

Then it follows that there exists a $\delta_1 > 0$ s.t. if $a < x < a + \delta_1$ then $|f(x) - f(a)| < -f(a)$, that is, $f(x) < 0$
V. By the same idea and assumption f is left continuous at b. We take

$$\epsilon = f(b) > 0$$

then it follows that there is such a δ_2 such that if $b - \delta_2 < x < b$ then $|f(x) - f(b)| < f(b)$, i.e., $f(x) > 0$
VI. Note the fact that S is bounded above by b by assumption and $a + \delta \in S$.
Thus, S is non-empty
VII. By the completeness property of $\mathbb{R}$ S has a least upper bound, say $c = \sup(S)$

9.2 Yet finished

10 Lec 8b

10.1 Derivatives

10.2 Differentiability

Theorem 8.2.1: If f is differentiable at a, then f is continuous at a.
Counterexample against the converse of the theorem: Consider $f(x) = |x|$ which is a continuous function on Domain: $\mathbb{R}$ however it is not differentiable at $x = 0$. Then f is continuous at $x = 0$ but not differentiable at $x = 0$

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|0+h| - f|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

When $h \to 0^-$, $\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = -1$.
When $h \to 0^+$, $\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = 1$.

10.3 Derivative notation

$$f^{(n)}(x) \equiv y^{(n)} \equiv \frac{d^n f}{dx^n} \equiv \frac{d^n y}{dx^n}.$$

Also, note that $\frac{d}{dx}$ is called the derivative operator.
Example for n^{th} derivative:

$$\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\cdots f\right)\right)$$

Theorem 8.3.1: Let $a \in \mathbb{R}$ and let f and g be functions defined in an interval around a.
If $f \wedge g$ are differentiable at a, then $f + g$ is differentiable at a, and furthermore,

$$(f + g)'(a) = f'(a) + g'(a).$$

10.4 Derivative Rules

$$(c)' = 0$$
$$(cx)' = c$$
$$(x^n)' = nx^{n-1}$$
$$(cf)' = cf'$$
$$(f \pm g)' = f' \pm g'$$
$$(fg)' = f'g + fg'$$
$$(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$$
$$(\sin x)' = \cos x$$
$$(\cos x)' = -\sin x$$
$$(\tan x)' = -\sec^2 x$$
$$(\sqrt{x})' = -\frac{1}{2\sqrt{x}}$$
$$(\frac{1}{x})' = \frac{-1}{x^2}$$

Theorem 8.4.1: Let $c \in \mathbb{R}$. Then $\frac{d}{dx}c = 0$
Proof.
Let $f(x) = c$
Then for any $c \in \mathbb{R}$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} && \text{by derivative definition} \\
&= \lim_{h \to 0} \frac{c - c}{h} && \text{by definition of } f \\
&= \lim_{h \to 0} \frac{0}{h} && \text{by algebra} \\
&= \lim_{h \to 0} 0 && \text{by cancellation theorem} \\
&= 0 && \text{by limit rules.}
\end{aligned}$$

Theorem 8.4.2: If n is a positive integer, then $\frac{d}{dx}x^n = nx^{n-1}$

Proof.

Let $f(x) = x^n$ be a power function for some positive integer n.

We use the formula:

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$$

Then at any number $a \in \mathbb{R}$ we have:

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x-a} \\
&= \lim_{x\to a} \frac{x^n - a^n}{x-a} \\
&= \lim_{x\to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x-a} \\
&= \lim_{x\to a}(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) && \text{by Cancellation Theorem} \\
&= nx^{n-1} && \text{by limit rule and continuity}
\end{aligned}$$

Theorem 8.4.3: Let $a \in \mathbb{R}$ and suppose f, g are functions defined on an interval around a.
Define $h = f \cdot g$
If f and g are differentiable at a, then h is differentiable at a, and furthermore

$$(fg)'(a) = h'(a) = f'(a)g(a) + f(a)g'(a).$$

Proof.
Assume f and g are differentiable at a

i.e., by definition $\begin{cases} \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} \text{exists} \\ \lim\limits_{x \to a} \dfrac{g(x) - g(a)}{x - a} \text{exists} \end{cases}$

Then,

$$\begin{aligned}
h'(a) &= \lim_{x \to a} \frac{h(x) - h(a)}{x - a} && \text{by definition of derivative} \\
&= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a} && \text{by definition of h} \\
&= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x - a} && \text{by algebra} \\
&= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \cdot g(x) + f(a) \cdot \frac{g(x) - g(a)}{x - a} \right] && \text{by algebra} \\
&= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} g(x) + \lim_{x \to a} f(a) \cdot \lim_{x \to a} \frac{g(x) - g(a)}{x - a} && \text{by algebra}
\end{aligned}$$

note that $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists since f is differentiable at a;

$\lim\limits_{x \to a} g(x)$ exists since g is differentiable at a, hence g continuous at a by theorem (diff $\implies$ cont);

$\lim\limits_{x \to a} f(a)$ exists by constant limit rule;

$\lim\limits_{x \to a} \dfrac{g(x) - g(a)}{x - a}$ exists since g is differentiable at a.

$$= f'(a)g(a) + f(a)g'(a) \qquad \text{as required.}$$

□

Theorem 8.4.4:

$$(\sin x)' = \cos x$$

Proof.
Let $f(x) = \sin x$
Then,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} && \text{by definition of derivative}\\
&= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} && \text{by definition of } f\\
&= \lim_{h\to 0} \frac{[\sin x\cos h + \cos x \sin h] - \sin x}{h} && \text{by trigonometric identity}\\
&= \lim_{h\to 0} \frac{[\sin x\cos h - \sin x] + \cos x \sin h}{h} && \text{by algebra}\\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0}\frac{\cos x\sin h}{h} && \text{by limit rules}\\
&= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h} && \text{by limit rules}\\
&= \sin x \cdot 0 + \cos x \cdot 1 && \text{by property } \lim_{x\to 0}\tfrac{\cos x - 1}{x} = 0 \text{ and } \lim_{x\to 0}\tfrac{\sin x}{x} = 1\\
&= \cos x && \text{by algebra; as required.}
\end{aligned}$$

□

Theorem 8.4.5:

$$(\cos x)' = -\sin x$$

Proof.
Let $f(x) = \cos x$
Then,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} && \text{by definition of derivative}\\
&= \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} && \text{by definition of } f\\
&= \lim_{h\to 0} \frac{[\cos x\cos h - \sin x \sin h] - \cos x}{h} && \text{by trigonometric identity}\\
&= \lim_{h\to 0} \frac{[\cos x\cos h - \cos x] - \sin x \sin h}{h} && \text{by algebra}\\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0}\frac{\sin x\sin h}{h} && \text{by limit rules}\\
&= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0}\frac{\sin h}{h} && \text{by limit rules}\\
&= \cos x \cdot 0 - \sin x \cdot 1 && \text{by property } \lim_{x\to 0}\tfrac{\cos x - 1}{x} = 0 \text{ and } \lim_{x\to 0}\tfrac{\sin x}{x} = 1\\
&= -\sin x && \text{by algebra; as required.}
\end{aligned}$$

□

Example: There are two tangent lines to the graph of $f(x) = x^2 - 2x + 1$ that are also tangent to $g(x) = -x^2 - 2x$. Determine the slopes of these lines.

<u>Solution:</u> Consider

$$\begin{cases} m = f'(a) = 2a - 2 \\ m = g'(b) = -2b - 2 \\ m = \frac{f(a)-g(b)}{a-b} = \frac{(a^2-2a+1)-(-b^2-2b)}{a-b} \end{cases}$$

Where

$$a = \frac{m+2}{2} \text{ and } b = \frac{-m-2}{2}$$

by algebra.

Then, we have

$$\begin{aligned} m = \frac{(a^2-2a+1)-(-b^2-2b)}{a-b} &= \frac{((\frac{m+2}{2})^2 - 2(\frac{m+2}{2}) + 1) - (-(\frac{-m-2}{2})^2 - 2(\frac{-m-2}{2}))}{(\frac{m+2}{2}) - (\frac{-m-2}{2})} && \text{by algebra} \\ &= \frac{m^2-2}{2(m+2)} && \text{by algebra} \\ \implies 0 &= \frac{m^2-2}{2(m+2)} - m && \text{by algebra} \\ &= \frac{m^2-2-m(2(m+2))}{2(m+2)} && \text{by algebra} \\ &= m^2 - 2 - 2m^2 - 4m && \text{by algebra} \\ &= m^2 + 4m + 2 && \text{by algebra} \\ \implies m &= \frac{-4 \pm 2\sqrt{2}}{2} && \text{by quadratic formula} \\ &= -2 \pm \sqrt{2} && \text{by algebra.} \end{aligned}$$

Therefore the slopes are $-2 \pm \sqrt{2}$ as required. □

Theorem 8.4.6: Let $a \in \mathbb{R}$ and suppose $f, g$are functions defined on an interval around a.
Define $h = \dfrac{f}{g}$
If f and g are differentiable at a, then h is differentiable at a, and furthermore

$$(\frac{f}{g})'(a) = h'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

Assume f, g, are differentiable at a
i.e., by definition $\begin{cases} \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a} \text{exists} \\ \lim\limits_{x\to a} \dfrac{g(x)-g(a)}{x-a} \text{exists.} \end{cases}$
Then,

$$\begin{aligned}
h'(a) &= \lim_{x\to a} \frac{h(x)-h(a)}{x-a} && \text{by definition of derivative} \\
&= \lim_{x\to a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} && \text{by definition of } h \\
&= \lim_{x\to a} \frac{\frac{f(x)g(a)-f(a)g(x)}{g(x)g(a)}}{x-a} && \text{by algebra} \\
&= \lim_{x\to a} \frac{f(x)g(a)-f(a)g(x)}{(x-a)g(x)g(a)} && \text{by algebra} \\
&= \lim_{x\to a} \frac{f(x)g(a)-f(x)g(x)+f(x)g(x)-f(a)g(x)}{(x-a)g(x)g(a)} && \text{by algebra} \\
&= \lim_{x\to a} \left[\frac{g(x)-g(a)}{x-a} \cdot \frac{-f(x)}{g(x)g(a)} + \frac{f(x)-f(a)}{x-a} \cdot \frac{g(x)}{g(a)g(x)}\right] && \text{by algebra} \\
&= \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \cdot \lim_{x\to a} \frac{-f(x)}{g(x)g(a)} + \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a} \frac{g(x)}{g(a)g(x)}
\end{aligned}$$

since f, g is differentiable so does $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ and $\lim\limits_{x\to a} \dfrac{g(x)-g(a)}{x-a}$ exist;

$\lim\limits_{x\to a} \dfrac{-f(x)}{g(x)g(a)}$ also exists as differentiability $\implies$ continuity and limit constant rule

$\lim\limits_{x\to a} \dfrac{g(x)}{g(a)g(x)}$ exists since g is differentiable hence continuous and $g(a)$ by limit constant rule

$$\begin{aligned}
&= g'(a) \cdot \frac{-f(a)}{g(a)^2} + f'(a) \cdot \frac{g(a)}{g(a)^2} && \text{apply limit rules} \\
&= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2} && \text{as required. } \blacksquare
\end{aligned}$$

Practice Problem: Let $f(x) = x^{10} + 4x + 8$
Then

$$\begin{aligned}
\lim_{w\to 2} \frac{f'(w) - f'(2)}{w-2} &= \lim_{w\to 2} \frac{(10w^9 + 4) - (10(2)^9 + 4)}{w-2} && \text{by definition of derivative and } f \\
&= \lim_{w\to 2} \frac{(10w^9 - 10(2)^9}{w-2} && \text{by algebra} \\
&= \lim_{w\to 2} \frac{10(w^9 - (2)^9)}{w-2} && \text{by algebra} \\
&= \lim_{w\to 2} \frac{10((w^3)^3 - ((2)^3)^3)}{w-2} && \text{by algebra} \\
&= \lim_{w\to 2} \frac{10(w^3 - 2^3)((w^3)^2 + w^3 2^3 + (2^3)^2)}{w-2} && \text{by identity } x^3 - y^3 = (x-y)(x^2 + xy + y^2) \\
&= \lim_{w\to 2} \frac{10(w-2)(w^2 + 2w + 2^2)(w^6 + w^3 2^3 + 2^6)}{w-2} \\
& && \text{by identity } x^3 - y^3 = (x-y)(x^2 + xy + y^2) \\
&= \lim_{w\to 2} 10(w^2 + 2w + 2^2)(w^6 + w^3 2^3 + 2^6) && \text{by cancellation theorem} \\
&= 10(2^2 + 2^2 + 2^2)(2^6 + 2^3 2^3 + 2^6) && \text{by limit rules} \\
&= 10 \cdot 3^2 \cdot 2^8 && \text{by limit rules} \\
&= 23040 && \text{as required } \square.
\end{aligned}$$

Term Test II Review

1. **Squeeze Theorem**: Its proof and variations (e.g., see Problem 5.2 in Practice Problem Set 5).
   Let $L, c \in \mathbb{R} \land \exists d > 0$ s.t. f, g, h are functions defined on a punctured interval around c. If $\forall x \in (c - d, c) \cup (c, c+d), g(x) \leq f(x) \leq h(x)$ and $\lim_{x \to c} g(x) = \lim_{x \to c} h(x)$
   then,
   $$\lim_{x \to c} f(x) = L$$
   Proof.
   Assume $\forall x \in (c - d, c) \cup (c, c+d), g(x) \leq f(x) \leq h(x)$ and $\lim_{x \to c} g(x) = \lim_{x \to c} h(x)$
   WTS:
   $$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$
   Let $\epsilon > 0$ be arbitrary. By assumption we have both
   $$\begin{cases} \forall \varepsilon_1 > 0 \exists \delta_1 > 0 \text{ s.t. } 0 < |x - c| < \delta_1 \implies |g(x) - L| < \epsilon_1 \\ \forall \varepsilon_2 > 0 \exists \delta_2 > 0 \text{ s.t. } 0 < |x - c| < \delta_2 \implies |h(x) - L| < \epsilon_2 \end{cases}$$
   Let $\varepsilon_1 = \land \varepsilon_2 = \varepsilon$
   Choose $\delta = \min\{\delta_1, \delta_2, d\}$
   Then it follows that $0 < |x - c| < \delta_1, 0 < |x - c| < \delta_2$, and $0 < |x - c| < d$, which suffice for $|g(x) - L| < \epsilon_1, |h(x) - L| < \epsilon_2, g(x) \leq f(x) \leq h(x)$.
   Then,
   $$\begin{aligned} & L - \varepsilon < g(x) < L + \varepsilon \land L - \varepsilon < h(x) < L + \varepsilon \land g(x) \leq f(x) \leq h(x) \\ & \implies L - \varepsilon < g(x) \leq f(x) \leq h(x) < L + \varepsilon \\ & \implies |f(x) - L| < \varepsilon \\ & \text{i.e., } \lim_{x \to c} f(x) = L \end{aligned}$$
   ■

2. **Limits involving trigonometric functions**: Proofs and examples such as:

$$\lim_{x\to 0} \sin x = 0, \quad \lim_{x\to 0} \cos x = 1, \quad \lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0.$$

$$\lim_{x\to 0} \sin x = 0$$

<u>Proof.</u>
WTS: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x| < \delta \implies |\sin x| < \delta$
Let $\varepsilon > 0$ be arbitrary
Choose $\delta = \varepsilon$
We use the inequality $|\sin x| \leq |x|$
On a unit circle, this is always true as the arch of angel x is greater or equal than the height $\sin(x)$ of the triangle where $\theta = x$.
Assume $0 < |x| < \delta = \epsilon$ by our chosen δ
Then, $|\sin x| \leq |x| < \delta = \epsilon$ as required. ■

$$\lim_{x\to 0} \cos x = 1$$

<u>Proof.</u>
WTS: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x| < \delta \implies |\cos x - 1| < \delta$
Let $\varepsilon > 0$ be arbitrary.
Choose $\delta = 2\sqrt{\varepsilon}$. Note that $\delta > 0$
Assume $0 < |x| < \delta$.
Then, $|\frac{x^2}{4}| < \frac{\delta^2}{4}$
Then,

$$\begin{aligned} |\cos x - 1| &= |(1 - \sin^2(\frac{x}{2})) - 1| && \text{by double angel identity} \\ &= |\sin^2(\frac{x}{2})| \leq |\frac{x^2}{4}| && \text{by identity used in previous proof} \\ &< \frac{\delta^2}{4} \\ &= \frac{(2\sqrt{\varepsilon})^2}{4} \\ &= \frac{4\varepsilon}{4} = \varepsilon && \text{as required. ■} \end{aligned}$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Proof.
We prove by squeeze theorem using geometry.
Observe that on unit circle$\forall x\in[-\frac{\pi}{2},\frac{\pi}{2}]$,

$$\begin{aligned}\frac{x}{2\pi}\cdot\pi(1)^2&\le\frac{\tan x}{2}\\ \implies \frac{x}{2}&\le\frac{\tan x}{2}\\ \implies x&\le\tan x\\ \implies x\cos x&\le\sin x\\ \implies \cos x&\le\frac{\sin x}{x}\end{aligned}$$

Also the fact that $\forall x\in\mathbb{R}$,

$$\begin{aligned}&|\sin x|\le|x|\\ &\implies -x\le\sin x\le x\\ &\implies \sin x\le x\\ &\implies \frac{\sin x}{x}\le 1\end{aligned}$$

Combining these, we thus have,

$$\forall x\in[-\frac{\pi}{2},\frac{\pi}{2}],\cos x\le\frac{\sin x}{x}\le x$$

Note that

$$\lim_{x\to 0}\cos x=\lim_{x\to 0}1=1$$

By squeeze theorem,

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

as required. ■

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0$$

Proof.

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x}&=\lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}\\ &=\lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}\\ &=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}\\ &=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{(1+\cos x)}\\ &=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{1+\cos x}\\ &=1\cdot\frac{0}{2}=0\end{aligned}$$

as required. ■

3. **Continuity**: Both the definition using limits and the ε-δ definition, and how to apply these to problems.
   Recall the definition:
   A function is said to be continuous at a point c if

$$\forall \varepsilon > 0, \exists \delta \text{ s.t. } |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon \equiv \lim_{x \to c} f(x) = f(c)$$

   For left right continuity:
   We say a function f defined on an open interval $I := (a, b)$ is continuous,**if** it is continuous $\forall c \in (a, b)$ and right continuous at a and left continuous at b, i.e.,

$$\forall \varepsilon > 0, \exists \delta \text{ s.t. } x - a < \delta \implies |f(x) - f(a)| < \varepsilon \equiv \lim_{x \to a^+} f(x) = f(c)$$

$$\forall \varepsilon > 0, \exists \delta \text{ s.t. } -\delta < x - b \implies |f(x) - f(b)| < \varepsilon \equiv \lim_{x \to b^-} f(x) = f(b)$$

4. **The Continuity Theorem**: Partial proofs, including proving that the basic trigonometric function $\sin x$ is continuous and that the sum of continuous functions is continuous.

The Continuity Theorem

Theorem

If f is a function constructed using any finite combination of the following operations:

- sum, difference, product, quotient, and
- composition of functions,

where the component functions are

- polynomial and rational functions
- root functions,
- absolute value functions,
- exponential and logarithmic functions,
- trigonometric and inverse trigonometric functions,

then f is continuous on its domain.

Proof (Outline). We leave this as an exercise. Note that it is enough to show the following.
Prove the sum, product, quotient and composition of continuous functions is continuous.
Then prove some basic functions are continuous, for example:

(i) $f(x) = c$, where $c \in \mathbb{R}$	(iv) $f(x) = \ln x$	(vii) $f(x) = \sqrt{x}$
(ii) $f(x) = x$	(v) $f(x) = \sin x$	(viii) $f(x) = \|x\|$
(iii) $f(x) = e^x$	(vi) $f(x) = \cos x$	(ix) $f(x) = \arcsin x$

5. **The Cancellation Theorem**.

6. **Limit computations**.

7. **Definitions of max, min, sup, inf, upper/lower bounds, and boundedness above/below**, as well as the Least Upper Bound Property.
   Let $S \subseteq \mathbb{R}$ and $M, m, b, \ell, s, a \in \mathbb{R}$. Then:
   (a) M is the **maximum** of S if $M \in S$ and $x \leq M$ for all $x \in S$.
   (b) m is the **minimum** of S if $m \in S$ and $x \geq m$ for all $x \in S$.
   (c) b is an **upper bound** of S if $x \leq b$ for all $x \in S$.
   (d) ℓ is a **lower bound** of S if $x \geq \ell$ for all $x \in S$.
   (e) s is the **supremum** of S (denoted $s = \sup(S)$) if:
      i. s is an upper bound of S, and
      ii. if b is an upper bound of S, then $s \leq b$.
   (f) a is the **infimum** of S (denoted $a = \inf(S)$) if:
      i. a is a lower bound of S, and
      ii. if ℓ is a lower bound of S, then $a \geq \ell$.
   (g) S is **bounded above** if it has at least one upper bound.
   (h) S is **bounded below** if it has at least one lower bound.
   (i) S is **bounded** if it is both bounded above and bounded below.

8. **The Approximation Theorem for sup and its proof**:

$$s = \sup(S) \iff \forall \varepsilon > 0, \exists x \in S \text{ such that } s - \varepsilon < x.$$

   We prove both directions, first, $s = \sup(S) \implies \forall \varepsilon > 0, \exists x \in S$ such that $s - \varepsilon < x$.
   Assume $s = \sup(S)$.
   Let $\varepsilon > 0$ be arbitrary.
   Assume to the contrary $\forall x \in S, s - \varepsilon \geq x$.
   Then $s - \varepsilon$ is not an upper bound since s is the least upper bound.
   Thus $\exists x \in S$ s.t. $s - \varepsilon < x$ this contradicts our assumption.
   Then we prove $s = \sup(S) \impliedby \forall \varepsilon > 0, \exists x \in S$ such that $s - \varepsilon < x$.
   Assume $\forall \varepsilon > 0, \exists x \in S$ such that $s - \varepsilon < x$
   Assume to the contrary $s \neq \sup(S)$
   Then there is an upper bound $b < s$ since s is not an the supremum.
   Let $\varepsilon = s - b > 0$
   Then $\exists x \in S$ s.t. $s - (s - b) < x$,i.e., $\exists x \in S$ s.t. $b < x$.
   Thus b is not an upper bound contradicting our assumption.
   We have proven both directions as required. ■

9. **Archimedean Property and its proof**: Includes density of rational numbers and proof, as well as computing the infimum and supremum of sets.
   Proof.
   WTS $\forall x, y \in \mathbb{R}, x, y > 0 \implies \exists n \in \mathbb{N}$ s.t. $nx > y$
   Let x, y be arbitrary.
   Suppose $x, y > 0$.
   Suppose to the contrary $\forall n, nx \leq y$
   Consider a set $S = \{nx : n \in \mathbb{N}\}$
   Then, y is an upper bound by definition. And S is not an empty set.
   By completeness property, there is an supremum of S, say, $\alpha = \sup(S)$.
   Then, $\alpha - x$ is not a upper bound as $\alpha \leq$ any upper bound.
   In particular, by approximation theorem, take $s = \alpha$ and $\varepsilon = x$, $\exists n \in S$ s.t. $\alpha - x < nx$
   Thus for some n, $\alpha = nx + x + (n + 1)x$. Since $n \in N, (n + 1)x \in S$.This contradicts our assumption that α is a supremum.
   Therefore it must be the case that $\exists n \in \mathbb{N}$ s.t. $nx > y$ as required. ■

10. **The Boundedness Theorem and its proof** (omit the Extreme Value Theorem).
   Proof.
   We want to show that if f is continuous on $[a, b]$, then f is bounded on $[a, b]$. Formally, $\forall x \in [a, b], \exists M$ s.t. $|f(x)| \leq M$
   Assume f is continuous on $[a, b]$
   Consider a set $S = \{x \in [a, b] : f \text{ is bounded on } [a, x]\}$
   Then, $S \neq \emptyset$ and is bounded above by b.
   By Completeness Property of Reals, there is a least upper bound, say $c = \sup(S)$
   Note that this means any upper bound, including $b \geq c$. We claim that $c = b$
   Suppose $b > c$.
   Then by right continuity at a there is a δ s.t. f is bounded on $[a, a + \delta]$.
   Also since f is cont at c as $c < b$, there exists a δ s.t. f is bounded on $[c - \delta, c + \delta]$
   Which means f is bounded on $[a, c + \delta]$. This contradicts our assumption that $c = \sup(S)$.
   Thus we have f is bounded on $[a, x] \forall, a < b$. Since f is left continuous at c, there is a δ s.t. f is bounded on $[b - \delta, b]$. Combining these we have f is bounded on $[a, b]$. ■

11. **Intermediate Value Theorem**: Its proof and how to apply it to problems.
    Proof.
    We prove if f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$ then $\exists c \in (a, b)$ s.t. $f(c) = 0$.
    Assume f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$.
    Consider a set $\{\gamma \in [a, b] : f \text{ is negative on } [a, \gamma)\}$
    Since f is right continuous at a we know there is a δ such that $f(x) < 0$ is bounded on $(a, a + \delta)$ Thus $a + \delta \in S \implies S \neq \emptyset$. Also note that S is bounded above by b.
    By Completeness Property there is a supremum of S we call it $c = \sup(S)$.
    Claim 1: $c \neq a$ as $f(x) < 0 \forall x \in (a, \delta + a); c > a$.
    Claim 2: $c \neq b$. as $f(x) > 0, \forall x \in (b - \delta, b); c < b$
    Thus $c \in (a, b)$. Next we show that $f(c) = 0$.
    Suppose $f(c) > 0$, then the positive interval would extend to the left of c as we can always find a value in between $f(c) \wedge 0$ by density of R. Which is impossible as anything less than c should be in S.
    Suppose $f(c) < 0$, then there is some t s.t. $c < t \in [a, t)$ contradicting our assumption that c is the supreme.
    Therefore it must be the case that $f(c) = 0$ as required. ■

12. **Definition of the derivative, tangent line, and differentiability**.

13. Proof that **"differentiability implies continuity"**.

14. **Derivative rules and their proofs**: Constant multiple rule, sum/difference rules, product/quotient rules, and rules for trigonometric functions.

15. **The Chain Rule** (omit the proof).

16. **Derivative computations using derivative rules**.

Final Exam Theorems and Review

1.Extreme Value Theorem

If f is continuous on $[a, b]$, then f obtain both absolute max and min on $[a, b]$.
Proof.
Assume $f(x)$ is continuous on $[a, b]$.
Then, by Boundness Theorem f is bounded i.e., $\forall x \in [a, b] \exists M$ s.t. $|f(x)| \leq M$
(since f is continuous on $[a, b]$)
Thus, $\{f(x)|x \in [a, b]\}$ is bounded.
Let $M = \sup\{f(x)|x \in [a, b]\}$. (such sup exists by LUB property)
We claim that M is the max; i.e., $\exists c \in [a, b]$ s.t. $f(c) = M$.
To derive a contradiction, suppose there isn't such c.
Then, by definition of supremum, $f(x) < M, \forall x \in [a, b]$.
Now, we define a function $g(x) = \frac{1}{M-f(x)}$ note that such function is continuous since f is continuous and its sum with a constant is continuous.
Also, note that per our contrary assumption, $f(x) < M \implies 0 < f(x) - M \implies g(x) > 0$.
Sine g is continuous on $[a, b]$,g is also bounded on $[a, b]$ by Boundness Theorem.
Thus, there exists some K such that $|g(x)| \leq K \implies -K \leq g(x) \leq K$.
By definition of g this means $\frac{1}{M-f(x)} \leq K$ which implies that $M - f(x) \geq K$.
Furthermore, $f(x) \leq M - K$. Recall the definition of supremum that for any supremum s, $\forall \epsilon > 0, \exists x \in S$ s.t. $s - \epsilon < x$. Take K as the epsilon, note that $K > 0$. Then this violates our definition that M is a supremum.
Therefore, it must be that M is a max. The proof of the attainment of a min follows similarly. ■.

0.1. Local and Absolute Extrema

1. f obtains local max at c if for a open interval I containing c, $\forall x \in I, f(x) \leq f(c)$
2. f obtains local min at c if for a open interval I containing c, $\forall x \in I, f(x) \geq f(c)$
3. f obtains absolute max at c if, $\forall x \in D, f(x) \leq f(c)$
4. f obtains absolute min at c if, $\forall x \in D, f(x) \geq f(c)$

0.1.1. Corollary

If f obtains global extrema at $x = c$ on (a, b), then f obtains local extrema at $x = c$.
We will be using this fact to prove Roll's Theorem.

1.1. Fermat's Theorem

If f has a local extrema at c and f is differentiable at c, then $\exists c$ s.t. $f'(c) = 0$.
Proof.
Assume f has a local extrema at c and f is differentiable at c. We prove by cases.
Note that f thus either has a local max or min at c.
First, suppose f obtains a local max at c.
Then, by definition, for an open interval I containing c, $\forall x \in I, f(x) \leq f(c)$.
Let $x \in I$ be arbitrary.
We know also, $f'(c)$ exists since f is differentiable at c, i.e., $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists.
It then follows that $\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \wedge \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c}$ exist. Note that as $x \to c^+$, $x - c > 0 \wedge x \to c^-, x - c < 0$. Also, observe that $f(x) \leq f(c) \implies f(x) - f(c) \leq 0$.
We thus have $\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \leq 0 \wedge \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \geq 0$
By Squeeze theorem, we have $\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \leq 0 \leq \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c}$
and $\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c}$, which means $f'(c) = 0$ as per required.
For another case where f has a local min at c the proof follows similarly. ■

1.1.1. Roll's Theorem

If $\begin{cases} (1) f \text{ continuous on } [a, b] \\ (2) f \text{ differentiable on } (a, b) \\ (3) f(a) = f(b) \end{cases}$ then, $\exists c \in (a, b)$ s.t. $f'(c) = 0$

Proof. Assume $(1, 2, 3)$ hold.
Suppose there is some constant k such that $f(a) = f(b) = k$. We consider three cases.
Case 1: suppose $\forall x, f(x) = k$, i.e., $f(x)$ is constant function.
Then it follows that $\forall x, f'(x) = 0$.
Thus, $\exists c \in (a, b)$ s.t. $f'(c) = 0$.
Case 2: Suppose $\exists x \in (a, b)$ s.t. $f(x) > k$.
By Extreme Value Theorem, there is a global max, say at $x = c \in [a, b]$ since $f(x)$ is continuous on $[a, b]$.
Thus by definition, $\forall x \in [a, b], f(x) \leq f(c)$.
Note that it follows for some x $k < f(x) \leq f(c)$ i.e., $f(c) > k$. Thus, $c \neq a, b$. In particular, $c \in (a, b)$
Now for a subset open interval of (a, b) containing c, say I, $\forall x \in I, f(x) \leq f(c)$, i.e., c is a local max and differentiable (by assumption (2)).
(Recall Coroallary 0.1.1. that global extrema obtained other than end points implies that it is also a local extrema.)
This suffices for Fermats theorem that $f'(c) = 0$.
I.e., $\exists c \in (a, b)$ s.t. $f'(c) = 0$ as required.
For Case 3, it follows similarly from Case 2 when we suppose $\exists x \in (a, b)$ s.t. $f(x) < k$ and then apply Extreme value theorem for an existence of a global min c that does not equal to *anorb*. Which is thus also a differentiable local extrema, by Fermat's Theorem, such c guarantees $f'(c) = 0$. As required. ■

1.1.1.1 Mean Value Theorem

If f is continuous on $[a,b]$ and differentiable on (a,b), then $\exists c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.
0.2 Lemma: The secant line of $f(x)$ connecting a, b is $y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$.
This is derived from definition of a secant line:

$$\begin{aligned} m &= \frac{y-y_1}{x-x_1} \\ &\implies m(x-x_1) = y - y_1 \\ &\implies y = m(x-x_1) + y_1 \\ &\implies y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a) \quad \text{in this case } m = \tfrac{f(b)-f(a)}{b-a}; x_1 = a \wedge y_1 = f(a); \text{thus as required.} \ \square \end{aligned}$$

Now, lets proceed to prove MVT.
Proof.
Assume f is continuous on $[a,b]$ and differentiable on (a,b).
We define $g(x) = f(x) -$ the secant line of a, b :: the "distance" between f and such secant line.
That is,

$$g(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right].$$

Note that g is continuous on $[a,b]$ by continuity theorem as it is the difference between some continuous function f and polynomial; also, g diff on (a,b) as polynomial diff on $\mathbb{R}$,thus , (a,b), and g is the difference of a diff function and polynomial.
By definition,

$$g(a) = f(a) - \left[\frac{f(b)-f(a)}{b-a}(a-a) + f(a)\right] = f(a) - f(a) = 0 \tag{1}$$

$$g(b) = f(b) - \left[\frac{f(b)-f(a)}{b-a}(b-a) + f(a)\right] = f(b) - f(b) = 0 \tag{2}$$

$$\implies g(a) = g(b) = 0 \qquad \text{by (1,2)}$$

Then, we have g is continuous on $[a,b]$, diff on (a,b), and $g(a) = g(b)$, which suffice for Roll's Theorem that

$$\exists c \in (a,b) \text{ s.t. } g'(c) = 0$$

We first observe that, $g'(x) = f(x) - \frac{f(b)-f(a)}{b-a}$.
Thus, c satisfies

$$\begin{aligned} g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \\ \implies f'(c) = \frac{f(b)-f(a)}{b-a} \end{aligned}$$

That is, $\exists c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$ as required. ■

2.1. Inverse function Theorem

If $f \wedge f^{-1}$ are differentiable inverse function, for appropriate value of x we have,

$$(f^{-1})'x = \frac{1}{f'(f^{-1}(x))}.$$

Proof.
By Cancellation Rules, we have

$$\begin{aligned} f(f^{-1}(x)) &= x, \forall x \in \text{Range} \\ &\implies \frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx} x \\ &\implies f'(f^{-1}(x)) \cdot (f^{-1})'x = 1 \\ &\implies (f^{-1})'x = \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

as required ■

2.2 Derivatives of inverse Trignometry

2.2.1 arcsine:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

Let $\sin x$ be restricted to $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$
Proof.

$$\begin{aligned}\frac{d}{dx}\arcsin x &= \frac{1}{(\sin'(\arcsin x))} \\ &= \frac{1}{\cos(\arcsin x)} \\ &= \frac{1}{\sqrt{1-x^2}} && \text{by triangle method; } \square\end{aligned}$$

2.2.2 arccos:

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

Let $\cos x$ be restricted to $x \in (0, \pi)$
Proof.

$$\begin{aligned}\frac{d}{dx}\arccos x &= \frac{1}{(\cos'(x))} \\ &= \frac{1}{-\sin(\arccos x)} \\ &= -\frac{1}{\sqrt{1-x^2}} && \text{by triangle method; } \square\end{aligned}$$

2.2.3 arctan:

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Let $\tan x$ be restricted to $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$
Proof.

$$\begin{aligned}\frac{d}{dx}\arctan x &= \frac{1}{(\tan'(\arctan x))} \\ &= \frac{1}{\sec^2(\arctan x)} \\ &= \frac{1}{1+\tan^2(\arctan x)} \\ &= \frac{1}{1+(\tan(\arctan x))^2} \\ &= \frac{1}{1+x^2} && \text{by triangle method; } \square\end{aligned}$$

2.2.4 arcsec:

$$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}, \forall |x| > 1$$

Let $\sec x$ be restricted to $x \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. arcsec has domain $(-\infty, 1) \cup (1,)$ Proof.

$$\begin{aligned}
\frac{d}{dx}\operatorname{arcsec} x &= \frac{1}{(\sec'(\operatorname{arcsec} x))} \\
&= \frac{1}{\sec(\operatorname{arcsec} x)\tan(\operatorname{arcsec} x)} \\
&= \frac{1}{x\tan(\operatorname{arcsec} x)} \\
&= \frac{1}{x\frac{x}{|x|}\sqrt{x^2-1}} \\
&= \frac{1}{|x|\sqrt{x^2-1}} && \text{by triangle method; } \square
\end{aligned}$$

2.3 Derivatives of exponential

2.3.1 e^x:

$$\frac{d}{dx}e^x = e^x \cdot \ln e \cdot (x)' = e^x$$

Proof.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} \\ &= \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h} \\ &= \lim_{h\to 0} \frac{e^x(e^h-1)}{h} \\ &= e^x \lim_{h\to 0} \frac{e^h-1}{h} && \lim_{h\to 0} \tfrac{e^h-1}{h} = 1 \\ &= e^x && \text{as required; } \square \end{aligned}$$

2.3.2 c^x for some $c \in \mathbb{R}$:

$$\frac{d}{dx}c^x = c^x \cdot \ln c \cdot (x)' = c^x \cdot \ln c$$

Proof.

$$\begin{aligned} \frac{d}{dx}c^x = \frac{d}{dx}e^{\ln(c^x)} &= \frac{d}{dx}e^{x\ln(c)} \\ &= e^{\ln(c^x)} \cdot (x'\ln(c) + x\ln(c)') \\ &= c^x \cdot \ln(c) && \text{as required; } \blacksquare \end{aligned}$$

2.3.4 $\ln x$:

$$\frac{d}{dx}\ln x = \frac{(x)'}{x} = \frac{1}{x}$$

Proof.
Let $y = \ln x \implies e^y = x$

$$\begin{aligned} \frac{d}{dx}e^y &= \frac{d}{dx}x \\ &\implies \frac{dy}{dx}e^y = 1 \\ &\implies \frac{dy}{dx} = \frac{1}{e^y} \\ &\implies \frac{dy}{dx} = \frac{1}{x} && \text{as required; } \square \end{aligned}$$

2.3.5 $\log_c x$ for some $c \in \mathbb{R}$ s.t. $c > 0 \land c \neq 1$:

$$\frac{d}{dx}\log_c x = \frac{(x)'}{(\ln c)x} = \frac{1}{(\ln c)x}$$

Proof.
Let $y = \log_c x \implies c^y = x$.

$$\begin{aligned}\frac{d}{dx}c^y &= \frac{d}{dx}x \\ &\implies \frac{dy}{dx}c^y \cdot \ln(c) = 1 \\ &\implies \frac{dy}{dx} = \frac{1}{x \cdot \ln(c)}\end{aligned}$$

as required. □

2.4. Hyperbolic function

2.4.0 Definitions

1. $\cosh^2 x - \sinh^2 x = 1$

2. $\cosh x = \frac{e^x + e^{-x}}{2}$; this is even

3. $\sinh x = \frac{e^x - e^{-x}}{2}$; this is odd

4. $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

5. $e^x = \cosh x + \sinh x$

Hyperbolic Identities

1. $\cosh^2 x - \sinh^2 x = 1$
2. $\sinh(-x) = -\sinh x$
3. $\cosh(-x) = \cosh x$
4. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
5. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

2.4.1 Inverses

2.4.1.1 $\operatorname{arccosh} x = \ln(x + \sqrt{x^2 - 1})$

Proof.

we solve for $x = \frac{e^y + e^{-y}}{2}$.

$$\begin{aligned} x &= \frac{e^y + e^{-y}}{2} \\ &\implies 2x = e^y + e^{-y} \\ &\implies 2xe^y = e^{2y} + 1 \\ &\implies e^{2y} - 2xe^y + 1 = 0 \\ &\implies e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} \\ &\implies e^y = \frac{2x \pm 2\sqrt{x^2 - 1}}{2} = x \pm \sqrt{x^2 - 1} \\ &\implies y = \ln\left(x + \sqrt{x^2 - 1}\right) && \text{since the domain of } \ln > 0 \end{aligned}$$

2.4.1.2 $\operatorname{arcsinh} x = \ln(x + \sqrt{x^2 + 1})$

Proof.

we solve for $x = \frac{e^y - e^{-y}}{2}$.

$$\begin{aligned} x &= \frac{e^y - e^{-y}}{2} \\ &\implies 2x = e^y - e^{-y} \\ &\implies 2xe^y = e^{2y} - 1 \\ &\implies e^{2y} - 2xe^y - 1 = 0 \\ &\implies e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} \\ &\implies e^y = \frac{2x \pm 2\sqrt{x^2 + 1}}{2} = x \pm \sqrt{x^2 + 1} \\ &\implies y = \ln\left(x + \sqrt{x^2 + 1}\right) && \text{since the domain of } \ln > 0 \end{aligned}$$

2.4.1.3 $\operatorname{arctanh} x = \frac{1}{2}\ln(\frac{1+x}{1-x})$

Proof.

we solve for $x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$.

$$\begin{aligned} x &= \frac{e^y - e^{-y}}{e^y + e^{-y}} \\ &\implies (e^y + e^{-y})x = e^y - e^{-y} \\ &\implies xe^y + xe^{-y} + e^{-y} - e^y = 0 \\ &\implies xe^{2y} + x + 1 - e^{2y} = 0 \\ &\implies (x-1)e^{2y} + x + 1 = 0 \\ &\implies e^{2y} = \frac{1+x}{1-x} \\ &\implies y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \end{aligned}$$

2.4.2 Derivatives

2.4.2.1 $\frac{d}{dx}\cosh x = \sinh x$

2.4.2.2 $\frac{d}{dx}\sinh x = \cosh x$

2.4.2.3 $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$

2.4.2.4 $\frac{d}{dx}\operatorname{arccosh} x = \frac{1}{\sqrt{x^2-1}}$

2.4.2.5 $\frac{d}{dx}\operatorname{arcsinh} x = \frac{1}{\sqrt{1+x^2}}$

2.4.2.6 $\frac{d}{dx}\operatorname{arctanh} x = \frac{1}{\sqrt{1-x^2}}$

3. Linearization, Optimization, Graph Sketching, L'Hôpital's rule

3.1 Linearization

We use the tangent line to approximate values of a function. The approximation at a point x using a nearby value a is:

$$f(x) \approx L(x) = f'(a)(x-a) + f(a)$$

3.2 Optimization

The objective is to find some absolute extrema.
In doing so, the optimization involves constraint.
From the constraint function we can express one variable in terms of another; we then substitute this variable back to our objective function.
Now, we state the domain of objective function. Thereafter we identify critical points, and then apply second derivative test.
(Note that Only One Critical Point in Town can be applied.) We otherwise test the end point and critical point onto the objective function and compare the outputs.

3.3 Graph Sketching

One prominent difficulty arises when we try to find the slant asymptote. To do so, see when $x \to \pm\infty$ what the function becomes. For example $f(x) = x + \sqrt{x^2+1} \approx 2x$

3.4 L'Hôpital's rule

Indeterminate form	Conditions	Transformation to $0/0$	Transformation to ∞/∞
$\frac{0}{0}$	$\lim_{x\to c} f(x) = 0,\ \lim_{x\to c} g(x) = 0$	—	$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{1/g(x)}{1/f(x)}$
$\frac{\infty}{\infty}$	$\lim_{x\to c} f(x) = \infty,\ \lim_{x\to c} g(x) = \infty$	$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{1/g(x)}{1/f(x)}$	—
$0\cdot\infty$	$\lim_{x\to c} f(x) = 0,\ \lim_{x\to c} g(x) = \infty$	$\lim_{x\to c} f(x)g(x) = \lim_{x\to c}\frac{f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)g(x) = \lim_{x\to c}\frac{g(x)}{1/f(x)}$
$\infty-\infty$	$\lim_{x\to c} f(x) = \infty,\ \lim_{x\to c} g(x) = \infty$	$\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c}\frac{1/g(x)-1/f(x)}{1/(f(x)g(x))}$	$\lim_{x\to c}(f(x)-g(x)) = \ln\lim_{x\to c}\frac{e^{f(x)}}{e^{g(x)}}$
0^0	$\lim_{x\to c} f(x) = 0^+, \lim_{x\to c} g(x) = 0$	$\lim_{x\to c} f(x)^{g(x)} = \exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$	$\lim_{x\to c} f(x)^{g(x)} = \exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$
1^∞	$\lim_{x\to c} f(x) = 1,\ \lim_{x\to c} g(x) = \infty$	$\lim_{x\to c} f(x)^{g(x)} = \exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)^{g(x)} = \exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$
∞^0	$\lim_{x\to c} f(x) = \infty,\ \lim_{x\to c} g(x) = 0$	$\lim_{x\to c} f(x)^{g(x)} = \exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$	$\lim_{x\to c} f(x)^{g(x)} = \exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$