MATA37

Calculus II for Mathematical Sciences

Lecture Notes

documented by Eric Wu

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Textbook: Calculus, Single Variable by Laura Taalman and Peter Kohn

Contents

0 Trigonometry Review

Theorem 0.0.0.1 (Sum Formula)**.**

$$
\begin{aligned}
\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) && (1)\\
\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) && (2)\\
\tan(\alpha+\beta) &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} && \text{by (1) and (2)}\\
&= \frac{\frac{\sin(\alpha)}{\cos(\alpha)}+\frac{\sin(\beta)}{\cos(\beta)}}{1-\frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} && \text{divide num and dem by } \cos(\alpha)\cos(\beta)\\
&= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}
\end{aligned}
$$

Corollary 0.0.0.1.1 (double angle formula)**.** Let $\alpha, \beta = \theta$. Then,

$$
\begin{aligned}
\sin(2\theta) &= \sin(\theta)\cos(\theta)+\sin(\theta)\cos(\theta) && \text{by (1)}\\
&= 2\sin(\theta)\cos(\theta)\\
\cos(2\theta) &= \cos(\theta)\cos(\theta)-\sin(\theta)\sin(\theta) && \text{by (2)}\\
&= \cos^2(\theta)-\sin^2(\theta)\\
&= 2\cos^2(\theta)-1 && \text{we call this *}\\
&= 1-2\sin^2(\theta) && \text{we call this **}\\
\tan(2\theta) &= \frac{2\tan(\theta)}{1-\tan^2(\theta)}
\end{aligned}
$$

Corollary 0.0.0.1.2. Thus, it also follows that,

$$
\begin{aligned}
\sin^2(\theta) &= \frac{1-\cos(2\theta)}{2} && \text{by **}\\
\cos^2(\theta) &= \frac{1+2\cos(2\theta)}{2} && \text{by *}\\
\tan^2(\theta) &= \sec^2(\theta)-1 && \text{from } \sin^2+\cos^2=1
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sin(\frac{\theta}{2}) &= \pm\sqrt{\frac{1-\cos(\theta)}{2}} \\
&= 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2}) \\
\cos(\frac{\theta}{2}) &= \pm\sqrt{\frac{1+\cos(\theta)}{2}} \\
&= \cos^2(\frac{\theta}{2}) - \sin^2(\frac{\theta}{2}).
\end{aligned}
$$

1 Definite Integral

1.1 Sums

Definition 1.1.1 (Sigma Notation)**.** Let $m, n, k \in \mathbb{Z}^{\geq 0}$ s.t. $m \leq k \leq n$. If a_k is a real-valued function of k, then

$$\begin{aligned} &a_m + a_{m+1} + \cdots + a_{n-1} + a_n \\ &= \sum_{k=m}^{n} a_k, \end{aligned}$$

where a_k is the *general term*, k is the *index*, m is the *initial value of index*, and n is *final value of index*.

Example. Express $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots - \frac{1}{666^2}$ in $\sum -$ notation. We first observe that the $a_k$has the for $\frac{1}{k^2}$. Now, to osccilates the sign $\pm$, we define $a_k = \frac{(-1)^{k+1}}{k^2}$. Note that $1 \leq k \leq 666$ in this series. We thus obtain:

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots - \frac{1}{666^2} = \sum_{k=1}^{666} \frac{(-1)^{k+1}}{k^2}.$$

Remarks 1.1.1.0.1. $\sum$ notation is *not* unique. Take for example the equivalence:

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} a_{k+1}.$$

Theorem 1.1.1.1 (Properties of Sigma Notation)**.** Let $n, k, l \in \mathbb{Z}^+$ s.t. $k \leq n$. If a_k and b_k are real-valued functions of k, then

i. $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$

ii. $\displaystyle\forall c \in \mathbb{R}, \sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k.$

iii. $\displaystyle\sum_{k=1}^{n} a_k = \sum_{k=1}^{l-1} a_k + \sum_{k=l}^{n} a_k$, where $1 < l \leq n$.

Properties (i and ii) are called *linearity property* of sigma notation.

Proof. It is sufficient to verify both (i. and ii.) by proving the following: If a_k and b_k are real-valued functions of k, then $\forall c \in \mathbb{R}, \sum_{k=1}^{n}(ca_k + b_k) = c\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$.
Suppose $a_k, b_k \in \mathbb{R}$. Let $c \in \mathbb{R}$ be arbitrary. Then,

$$\begin{aligned}
\sum_{k=1}^{n}(ca_k + b_k) &= (ca_1 + b_1) + (ca_2 + b_2) + \cdots + (ca_n + b_n) && \text{by } \Sigma \text{ definition} \\
&= (ca_1 + \ldots ca_n) + (b_1 + \cdots + b_n) \\
& && \text{by associativity and commutativity of reals under addition} \\
&= c(a_1 + \ldots a_n) + (b_1 + \cdots + b_n) && \text{left distributivity law} \\
&= c\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k && \text{by } \Sigma \text{ definition; as required.}
\end{aligned}$$

Furthermore, for (iii.), let $1 < l \leq n$. Then,

$$\begin{aligned}
\sum_{k=1}^{n} a_k &= a_1 + \cdots + a_n && \text{by } \Sigma \text{ definition} \\
&= a_1 + \cdots + a_{l-1} + a_l + a_{l+1} + \cdots + a_n \\
& && \text{since } 1 < l \implies 0 < l - 1 \implies l - 1 \in \mathbb{Z}^+ \text{ the equality holds} \\
&= (a_1 + \cdots + a_{l-1}) + (a_l + a_{l+1} + \cdots + a_n) && \text{by associativity} \\
&= \sum_{k=1}^{l-1} a_k + \sum_{k=l}^{n} a_k && \text{by } \Sigma \text{ definition; as required}
\end{aligned}$$

□

Example. Evaluate $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{5}{n^4}(k^3+1)$.

Solution. We first observe that $\frac{5}{n^4}$ is constant with respect to k. Then,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{5}{n^4}(k^3+1) &= \lim_{n\to\infty} \frac{5}{n^4} \sum_{k=1}^{n}(k^3+1) && \text{by } \Sigma \text{ property (ii.)} \\
&= \lim_{n\to\infty} \frac{5}{n^4} \left(\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 1 \right) && \text{by } \Sigma \text{ property (i.)} \\
&= \lim_{n\to\infty} \frac{5}{n^4} \left(\frac{n^2(n+1)^2}{4} + n \right) && \text{by proven formula in A67} \\
&= \lim_{n\to\infty} \frac{5}{4} \cdot \frac{(n+1)^2}{n^2} + \frac{5}{n^3} && \text{by algebra} \\
&= \lim_{n\to\infty} \frac{5}{4} \cdot \left(\frac{n+1}{n} \right)^2 + \frac{5}{n^3} && \text{by algebra} \\
&= \lim_{n\to\infty} \frac{5}{4} \cdot \left(1 + \frac{1}{n} \right)^2 + \frac{5}{n^3} && \text{by algebra} \\
&= \frac{5}{4} \cdot (1+0)^2 + 0 = \frac{5}{4}
\end{aligned}$$

by algebra and limit type $\frac{c}{\infty}$ for some constant $c \in \mathbb{R}$; as required.

■

1.2 Riemann Sums

Definition 1.2.1 (Partition)**.** Let $a, b \in \mathbb{R}, a < b$. A *partition* P of $[a, b]$ is a collection of a finite number of points in $[a, b]$, one of which is 'a' and another one is 'b'. We write $P = \{x_0, x_1, x_2, \ldots, x_n\}$ for some $n \in \mathbb{N}$ s.t. $a = x_0 < x_1 < \cdots < x_n = b$.

Example. Let $I = [0, 1]$. Then $P = \{\frac{1}{2}, 0, \frac{1}{5}, 1\}$ is a partition; where $x_1 = 0, x_1 = \frac{1}{2}, x_2 = \frac{1}{5}, x_3 = 1$.

Definition 1.2.2 (Riemann Partition)**.** Let $a, b \in \mathbb{R}, a < b$. Consider $I = [a, b]$. Then, a *Riemann partition* of I is a partition such that $x_i = a + i\Delta x$; where $\Delta x = \frac{b-a}{n}, i = 0, 1, \ldots, n$ for some $n \in \mathbb{N}$.

Example. Find the exact signed area A between $y = f(x) = e^x$ over $[0, 3]$. Say we have $n = 3$ for a Riemann partition.

In this case, $A \approx f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x = \sum_{i=1}^{3} f(x_{i-1})\Delta x$

Definition 1.2.3 (Riemann Sum)**.** Let $a, b \in \mathbb{R}, a < b$. Let $[a, b] \subseteq \text{dom}(f)$. Let $P = \{x_i\}_{i=0}^{n}$ be a Riemann partition of $[a, b]$. Then, a Riemann Sum for f on $[a, b] = \sum_{i=1}^{n} f(x_i^*)\Delta x$, for any $x_i^* \in [x_{i-1}, x_i]$ (sample points). In particular,
Left Riemann sum for f on $[a, b] :=$

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

Right Riemann sum for f on $[a, b] :=$

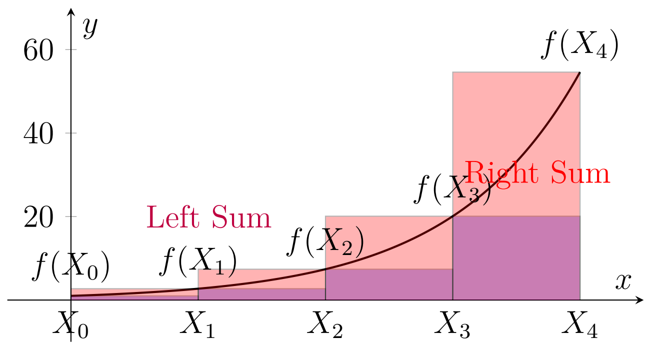
$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$

Conjecture.

1. (a) If f is increasing on $[a, b]$, then $L_n \leq A \leq R_n$.
   (b) If f is decreasing on $[a, b]$, then $R_n \leq A \leq L_n$.
2. Larger n values provide more accurate approximations of A.

Intuition. Let $y = f(x) = e^x$. Note that $f(x)$ is an increasing function on $\mathbb{R}$. Say $n = 5$. Then on the interval $[X_0, X_4]$ we have:

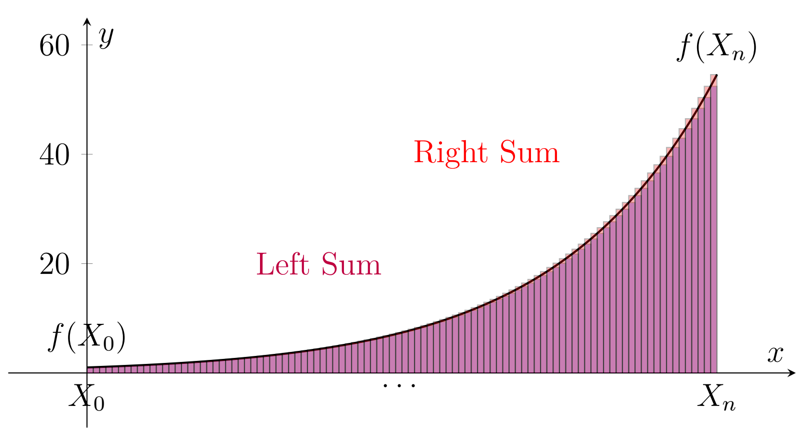
Left and Right Riemann Sum for e^x ($n = 5$)



Where $\Delta x = \frac{X_0 + X_4}{5}$ = all the base of rectangles. Observe that $L_n \leq A \leq R_n$ since left Riemann Sum takes $f(X_0), ..., f(X_3)$ and right Riemann Sum takes $f(X_1), ..., f(X_4)$ by our definition;where $f(x_0) < f(X_1)$ and $f(X_3) < f(X_4)$.
The signed area of this increasing function thus has the property satisfying conjecture 1., (a).

Consider $n \to \infty$ on the interval $[X_0, X_n]$of f.

Riemann Sum for e^x as $n \to \infty$



In this case, it seems that conjecture 2. holds.

1.3 Definite Integral

Definition 1.3.1 (Riemann Definite Integral)**.** Let $a, b \in \mathbb{R}, a < b$. Let A be signed area between f over $[a, b]$. Let a finite set $P = \{x_i\}_{i=0}^{n}$ be a Riemann partition of $[a, b]$.
Let $[a, b] \subseteq \operatorname{dom} f$. Then, the definite integral of f on $[a, b]$ is denoted as:

$$A = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

for any $x_i^* \in [x_{i-1}, x_i]$; provided that the limit exists.

Definition 1.3.2 (Integrability)**.** When such limit exists we say f is *integrable* on $[a, b]$.

Notation.

1. "$\int$" is called *integration sign*.
2. a and b are *integration limits*, where a is the *lower integration limit* and b is the *upper integration limit*.
3. $f(x)$ is the *integrand*.
4. dx is called the *differential*. (Intuitively the infinitesimally small width for Riemann Sum).

Example. Evaluate $\int_0^1 \sqrt{1-x^2}dx$.

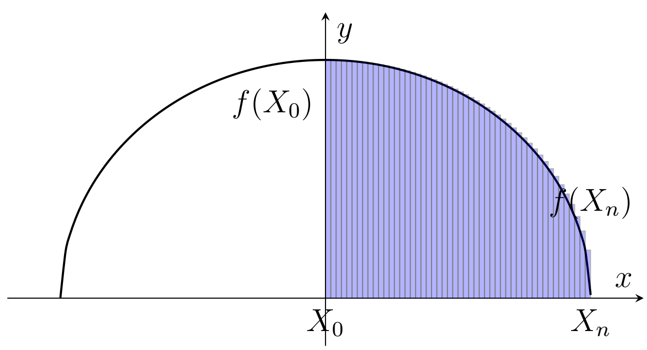
Solution. First, note that we have a smei-circle function:

$$\begin{aligned} f(x) = y = \sqrt{1-x^2} \\ \implies y^2 = 1-x^2 \\ \implies x^2+y^2 = 1 \quad & \text{observe that this is a circle center at } (0,0) \text{ with radius 1.} \end{aligned}$$

Since $\operatorname{dom} f = [1,-1] \implies f(x) \in [0,1]$ this is a semi-circle with radius 1. Geometrically, we thus have $\int_0^1 \sqrt{1-x^2}dx = \frac{\pi(1)^2}{4} = \frac{\pi}{4}$, as required. ■

Graphically,

Definite Integral $\int_0^1 \sqrt{1-x^2}dx$



Example.(Ans: $-\dfrac{21}{2}$). Compute $\int_0^3 (x-5)dx$ using Riemann definition of the definite integral. Given that $[a,b]=[0,3], f(x)=x-5$.

Solution. We know $\Delta x = \frac{3-0}{n} = \frac{3}{n}$. Thus, by definition, $x_i = 0 + i\frac{3}{n} = \frac{3i}{n}$. Choose $x_i^* = x_i$. Then,

$$f(x_i) = x_i - 5 = \frac{3i}{n} - 5.$$

Therefore,

$$\begin{aligned}
\int_0^3 (x-5)dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x && \text{by R definition of definite integral with } x_i^* = x_i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i}{n} - 5\right)\frac{3}{n} && \text{since } x_i = \tfrac{3i}{n} \text{ and } \Delta x = \tfrac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i}{n^2} - \frac{15}{n}\right) \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{9i}{n^2} - \sum_{i=1}^{n} \frac{15}{n}\right) && \text{property of } \Sigma \\
&= \lim_{n\to\infty} \left(\frac{1}{n^2}\sum_{i=1}^{n} 9i - \frac{1}{n}\sum_{i=1}^{n} 15\right) \\
&= \lim_{n\to\infty} \left(\frac{1}{n^2}(\frac{9n(n+1)}{2}) - \frac{1}{n}(15n)\right) \\
&= \lim_{n\to\infty} \left(\frac{9n^2+9}{2n^2} - 15\right) \\
&= -\frac{21}{2} && \text{as required.}
\end{aligned}$$

■

Example.(Ans: $\displaystyle\int_6^{11} \frac{x}{x-2}dx$). Express $\displaystyle\lim_{n\to\infty} \frac{5}{n}\sum_{i=1}^{n} \frac{6+\frac{5i}{n}}{\sqrt{4+\frac{5i}{n}}}$ as a definite integral.

Solution. We first note that

$$\lim_{n\to\infty} \frac{5}{n}\sum_{i=1}^{n} \frac{6+\frac{5i}{n}}{\sqrt{4+\frac{5i}{n}}} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{6+\frac{5i}{n}}{\sqrt{4+\frac{5i}{n}}} \cdot \frac{5}{n}$$

by properties of Σ.

Observe that $\frac{5}{n}$ seems to form a pattern. We thus choose $\Delta x = \frac{5}{n}$. Also, we choose $x_i^* = x_i$, i.e., to form a right Riemann sum. Then, by definition, $\Delta x = \frac{5}{n} = \frac{b-a}{n}$ and $x_i = a + i\Delta x_i$. In particular, we can choose $x_i = 6 + i\Delta x_i$. It then follows that $a = 6 \iff b = 11$. Then, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{5}{n}\sum_{i=1}^{n} \frac{6+\frac{5i}{n}}{\sqrt{4+\frac{5i}{n}}} &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{6+\frac{5i}{n}}{\sqrt{4+\frac{5i}{n}}} \cdot \frac{5}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i}{\sqrt{x_i-2}} \cdot \Delta x && \text{by chosen } x_i \text{ and } \Delta x
\end{aligned}$$

Note that $f(x_i^*) = \frac{x_i}{\sqrt{x_i-2}}$. We thus choose $f(x) = \frac{x}{\sqrt{x-2}}$. Then,

$$\begin{aligned}
\lim_{n\to\infty} \frac{5}{n}\sum_{i=1}^{n} \frac{6+\frac{5i}{n}}{\sqrt{4+\frac{5i}{n}}} &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i}{\sqrt{x_i-2}} \cdot \Delta x && \text{as shown previously} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \\
& \qquad \text{since } x_i^* = x_i \text{ as chosen, also by definition of chosen } f \\
&= \int_6^{11} f(x)dx && \text{R definition of definite integral} \\
&= \int_6^{11} \frac{x}{\sqrt{x-2}}dx && \text{as required.}
\end{aligned}$$

■

Theorem 1.3.2.1 (Properties of the Definite Integral)**.** Let $a, b \in \mathbb{R}, a < b$.
Let $[a, b] \subseteq \operatorname{dom} f$. If f and g are integrable on $[a, b]$, then

i. (a) If $f(x) \geq 0$ on $[a, b]$, then $\displaystyle\int_a^b f(x)dx \geq 0$.

   (b) If $f(x) \leq 0$ on $[a, b]$, then $\displaystyle\int_a^b f(x)dx \leq 0$.

ii. $f + g$ is integrable on $[a, b]$. Moreover,

$$\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

iii. $\forall c \in \mathbb{R}, cf$ is integrable on $[a, b]$. Furthermore,

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx.$$

iv. $\displaystyle\int_a^a f(x)dx = 0$.

v. $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx$

vi. Union Interval Property:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

for any constant $c \in (a, b)$.

Proof (of ii.). Suppose f and g are integrable on $[a, b]$. It is sufficient to show that $\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)dx + \int_a^b g(x)dx$. For the existence of $\int_a^b (f(x) + g(x))\, dx$ satisfies integrability of $f + g$ on $[a, b]$.
$*$ Let $P = \{x_i\}_{i=0}^n$ be a Riemann partition of $[a, b]$. Note that by assumption,

$$(1)\ \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x \qquad \text{exists for any } x_i^* \in [x_{i-1}, x_i]$$

$$(2)\ \int_a^b g(x)dx = \lim_{n\to\infty} \sum_{i=1}^n g(x_i^*)\Delta x \qquad \text{exists for any } x_i^* \in [x_{i-1}, x_i].$$

Consider

$$\begin{aligned}
\int_a^b f(x)dx + \int_a^b g(x)dx &= \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x + \lim_{n\to\infty} \sum_{i=1}^n g(x_i^*)\Delta x && \text{by hypothesis} \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^n f(x_i^*)\Delta x + \sum_{i=1}^n g(x_i^*)\Delta x \right) && \text{limit laws} \\
&= \lim_{n\to\infty} \sum_{i=1}^n \left(f(x_i^*)\Delta x + g(x_i^*)\Delta x \right) && \Sigma \text{ property} \\
&= \lim_{n\to\infty} \sum_{i=1}^n \left(f(x_i^*) + g(x_i^*) \right) \Delta x && \text{algebra} \\
&= \int_a^b (f(x) + g(x))dx && \text{by R definition of the definite integral}
\end{aligned}$$

because $\text{dom}(f + g) = \text{dom}(f) \cap \text{dom}(g)$
Moreorver, $[a, b] \subseteq \text{dom}(f)$ and $[a, b] \subseteq \text{dom}(g)$ by R hypothesis
it follows that $[a, b] \subseteq \text{dom}(f + g)$; as required.

□

Remarks 1.3.2.1.1 (Conjecture). Does $\int_0^1 f(x)dx$ exists? Where $f(x) = \begin{cases} 1\,, \text{ if } x \in \mathbb{Q} \\ 0\,, \text{ if } x \notin \mathbb{Q}. \end{cases}$

Let $a, b \in \mathbb{R}, a < b$. If f is continuous on $[a, b]$ or if f has a finite number of finite jump on $[a, b]$ then f is integrable on $[a, b]$.

1.4 The Darboux Integral

Definition 1.4.1 (Darboux Sums)**.** Let $a, b \in \mathbb{R}, a < b$. Let $P = \{x_i\}_{i=0}^{n}$ be *any* partition of $[a, b]$. Suppose f is bounded on $[a, b]$. That is, $\exists c \in \mathbb{R}^{\geq 0}, \forall x \in [a, b]$ s.t. $|f(x)| \leq c$. Furthermore note that it then follows for such c, $-c \leq f(x) \leq c$.

1. The Lower(Darbous) Sum for a function f with a partition P is denoted

$$L(f, P) := \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$.

2. The Upper (Darboux) Sum for a function f with a partition P is denoted

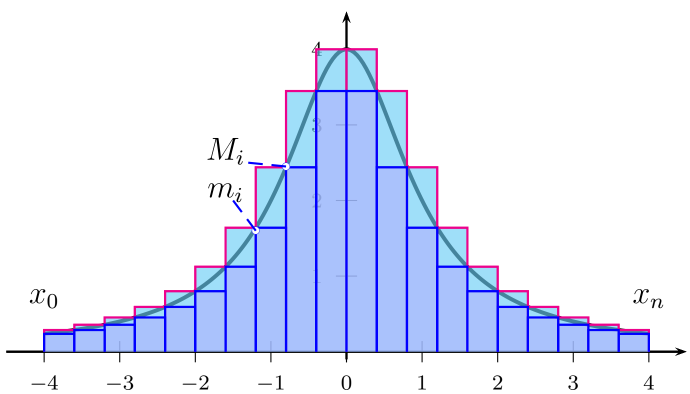
$$U(f, P) := \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$.

Definition 1.4.2 (Darboux Integral)**.** Let $a, b \in \mathbb{R}, a < b$. Suppose f is bounded on $[a, b]$. We say f is integrable on $[a, b]$, i.e., $\int_a^b f(x)dx$ exists, iff

$$\begin{aligned} &\sup\{L(f, p) : P \text{ is any partition of } [a, b]\} \\ =\, &\inf\{U(f, p) : P \text{ is any partition of } [a, b]\} \\ :=\, &\int_a^b f(x)dx \end{aligned}$$

Pictorially,



where the dark blue area is $L(f, P)$ and the light blue area is $U(f, P)$.

Example. Consider $g(x) = \begin{cases} 1\,, \text{if } x \in \mathbb{Q} \\ -1\,, \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
Compute $U(g, P)$ for any partition of $[0, 3]$.

Solution. Let $P = \{x_i\}_{i=0}^{n}$ be an arbitrary partition of $[0, 3]$.
For $i \in [1, n] \cap \mathbb{N}$,

$$\begin{aligned} M_i &= \sup\{g(x) : x \in [x_{i-1}, x_i]\} \\ &= \sup\{-1, 1\} \\ &\quad \text{Since } \mathbb{Q} \text{ and } \mathbb{R} \setminus \mathbb{Q} \text{ are } \textit{dense} \text{ in } \mathbb{R}, 1 \text{ and } -1 \text{ must be obtained by } x \in \mathbb{R}. \\ &= 1 && \text{by definition of sup.} \end{aligned}$$

Thus,

$$\begin{aligned} U(g, P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) && \text{by definition} \\ &= \sum_{i=1}^{n} (x_i - x_{i-1}) && \text{since } M_i = 1 \text{ as shown} \\ &= (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) && \text{by } \Sigma \text{ definition} \\ &= -x_0 + (x_1 - x_1 + \cdots + x_{n-1} - x_{n-1}) + x_n && \text{associativity and commutativity} \\ &= x_n - x_0 && \text{algebra} \\ &= 3 - 0 && \text{by definition of any partition over } [0, 3], a_0 = 0 \text{ and } a_n = 3 \\ &= 3 && \text{algebra as required.} \end{aligned}$$

■

Example. Let $f(x) = \begin{cases} 70+\pi\,, \text{if } x \in \mathbb{Q} \\ 0\,, \text{if } x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$
Prove f is not integrable on $[0,1]$ with Darboux definition.

Proof. WTS:

$$\sup\{L(f,p) : P \text{ is any partition of } [0,1]\} \neq \inf\{U(f,p) : P \text{ is any partition of } [0,1]\}.$$

Let $P = \{x_i\}_{i=0}^n$ for some $i \in [0,n]\cap\mathbb{N}$ and $n \in \mathbb{N}\cup\{0\}$ be arbitrary partition over $[0,1]$. Then, for each $i \in [1,n]\cap\mathbb{N}$,

$$\begin{aligned} m_i &= \inf\{f(x) : x \in [x_{i-1}, x_i]\} \\ &= \inf\{0, 70+\pi\} && \text{since } \mathbb{Q} \text{ and } \mathbb{R}\setminus\mathbb{Q} \text{ are dense in } \mathbb{R} \text{ both outputs are obtained} \\ &= 0 && \text{by definition of inf.} \end{aligned}$$

Furthermore,

$$\begin{aligned} M_i &= \sup\{f(x) : x \in [x_{i-1}, x_i]\} \\ &= \sup\{0, 70+\pi\} && \text{since } \mathbb{Q} \text{ and } \mathbb{R}\setminus\mathbb{Q} \text{ are dense in } \mathbb{R} \text{ both outputs are obtained} \\ &= 70+\pi && \text{by definition of sup.} \end{aligned}$$

It then follows that

$$\begin{aligned} U(f,P) &= \sum_{i=1}^n M_i(x_i - x_{i-1}) && \text{by definition of } U(f,p) \\ &= \sum_{i=1}^n (70+\pi)(x_i - x_{i-1}) && \text{by demonstrated fact that } M_i = 70+\pi \\ &= (70+\pi)\sum_{i=1}^n (x_i - x_{i-1}) && \Sigma \text{ linearity since } 70+\pi \text{ is constant} \\ &= (70+\pi)\cdot \text{length}([0,1]) && \text{by geometric intepretation} \\ &= (70+\pi)(1-0) && \text{definition of length} \\ &= 70+\pi && \text{algebra.} \end{aligned}$$

Also,

$$\begin{aligned}
L(f,P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) && \text{by definition of } L(f,p) \\
&= \sum_{i=1}^{n} 0(x_i - x_{i-1}) && \text{by demonstrated fact that } m_i = 0 \\
&= \sum_{i=1}^{n} 0 && \text{algebra} \\
&= 0 && \text{algebra.}
\end{aligned}$$

Thus, $\forall P$ of $[0,1], U(f,P) = 70 + \pi$ and $L(f,P) = 0$ since P is arbitrary.
Then,

$$\begin{aligned}
\sup\{L(f,P) : P \text{ is any partition of } [a,b]\} &= \sup\{0\} && \text{by demonstrated facts of } L(f,P) \\
&= 0 && \text{def of sup}
\end{aligned}$$

Also,

$$\begin{aligned}
\inf\{U(f,P) : P \text{ is any partition of } [a,b]\} &= \inf\{70+\pi\} && \text{by demonstrated facts of } U(f,P) \\
&= 70+\pi && \text{def of inf}
\end{aligned}$$

Therefore,

$$\sup\{L(f,p) : P \text{ is any partition of } [0,1]\} \neq \inf\{U(f,p) : P \text{ is any partition of } [0,1]\}.$$

That is, we have proven that f is not integrable on $[0,1]$ with Darboux definition as required.

$\square$

Definition 1.4.3 (Integrability Reformulation)**.** Let $a, b \in \mathbb{R}, a < b$. Suppose f is bounded on $[a, b]$. We say f is integrable on $[a, b]$, i.e., $\int_a^b f(x)dx$ exists, iff

$$\forall \varepsilon > 0, \exists P \text{ partition of } [a,b] \text{ s.t. } U(f,P) - L(f,P) < \varepsilon.$$

Example. Let $f(x) = \begin{cases} 1\,, \text{if } x \in \mathbb{Q} \\ 0\,, \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Prove $\displaystyle\int_0^1 f(x)dx$ DNE by Integrability Reformation.

Proof. WTS

$$\neg\left(\forall \varepsilon > 0, \exists P \text{ partition of } [0,1] \text{ s.t. } U(f,P) - L(f,P) < \varepsilon.\right) \text{ holds.}$$

That is,

$$\exists \varepsilon > 0, \forall P \text{ of } [0,1] \text{ s.t. } U(f,P) - L(f,P) \geq \varepsilon \text{ holds.}$$

Choose $\epsilon = \frac{e}{\pi} > 0$. Let $P = \{x_i\}_{i=0}^n$ be an arbitrary partition of $[0, 1]$.
Note that, for $i \in [1, n] \cap \mathbb{N}$,

$$\begin{aligned} m_i &= \inf\{f(x) : x \in [x_{i-1}, x_i]\} = 0 \\ M_i &= \sup\{f(x) : x \in [x_{i-1}, x_i]\} = 1. \end{aligned}$$

Now,

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^n M_i(x_i - x_{i-1}) - \sum_{i=1}^n m_i(x_i - x_{i-1}) && \text{by definition} \\ &= \sum_{i=1}^n (x_i - x_{i-1}) - \sum_{i=1}^n 0 \\ &= \text{length}([0,1]) - 0 \\ &= 1 \\ &> \frac{e}{\pi} && \text{by arithmetic fact} \\ &\geq \frac{e}{\pi} \\ &= \varepsilon && \text{by our choice of epsilon} \end{aligned}$$

Thus we have shown the required, that is, f is not integrable on $[0, 1]$. □

2 Indefinite Integral

Definition 2.0.1 (Antiderivative)**.** An antiderivative of a continuous function f over an interval I is a function F s.t.

$$\forall x \in I, F'(x) = f(x).$$

Example. Consider $f(x) = e^{x+1}$ on $[-2, 7]$.

Solution. Choose $F_1(x) = e^{x+1}$. Let $x \in [-2, 7]$ be arbitrary.
Then,

$$F_1'(x) = e^{x+1} \cdot (1) = e^{x+1} = f(x).$$

■

Excercise. Is $F_2(x) = e^{x+1} + 2$ also an antiderivative to f on I?

Solution. Yes. Since $F_2'(x) = e^{x+1} = f(x)$. ■

Remarks 2.0.1.0.1. Antiderivatives, when exist, are *unique up to an additive constant.*

Example. $f(x) = x^n$, s.t. $n \in \mathbb{R} \setminus \{-1\}$ on $I = (a, b), \forall a, b \in \mathbb{R}, a < b$.

Solution. Choose $F(x) = \frac{x^{n+1}}{n+1}$. Let $x \in I$ be arbitrary. Then,

$$F'(x) = \frac{1}{n+1}(x^{n+1})' = x^n = f(x).$$

■

Definition 2.0.2 (Indefinite Integral)**.** The indefinite integral of a continuous function f, denoted $\displaystyle\int f(x)dx$, is an infinite family of antiderivatives of f, i.e.,

$$\int f(x)dx = F(x) + C$$

where $F(x)$ is some antiderivative of f and C is an arbitrary constant.

Example. Evaluate $\displaystyle\int \frac{1}{4x^2+1}dx$.

Solution. Choose $F(x) = \frac{\arctan(2x)}{2}$. Then, $F'(x) = \frac{2}{1+4x^2} \cdot \frac{1}{2} = \frac{1}{4x^2+1}$. Thus, we have,

$$\begin{aligned}\int \frac{1}{4x^2+1}dx &= \int \frac{1}{(2x^2)+1}dx. \\ &= \frac{\arctan(2x)}{2} + C && \text{by chosen } F(x) \text{ as required.}\end{aligned}$$

■

Theorem 2.0.2.1 (Properties of Indefinite Integral)**.** If f and g are continuous, then

i. $\displaystyle\int (f(x)+g(x))dx = \int f(x)dx + \int g(x) + dx$.

ii. $\displaystyle\forall k \in \mathbb{R}, \int kf(x)dx = k\int f(x)dx$.

proof of (ii). Suppose f is continuous and $k \in \mathbb{R}$ is arbitrary. Consider

$$\begin{aligned}k\int f(x)dx &= k(F(x)+C) \\ & \text{by definition of indefinite integral of } f \text{ where } F'(x) = f(x), \forall x \in \operatorname{dom}(F) \\ &= kF(x) + kC && \text{by algebra}\end{aligned}$$

Now we claim that $(kF(x))' = kf(x)$. Let $x \in \operatorname{dom}(kf(x)) = \operatorname{dom}(f(x)$. Then,

$$(kF(x))' = kF'(x) = kf(x)$$

since $F'(x) = f(x)$ on $\operatorname{dom} F$. Thus, the claim holds. Thus,

$$\begin{aligned}k\int f(x)dx &= kF(x) + kC && \text{as shown previously} \\ &= kF(x) + \tilde{C} && \text{where } \tilde{C} = kC \text{ is some arbitrary constant} \\ &= \int kf(x)dx.\end{aligned}$$

□

Example. Find $\int \left(\frac{\sin(2x)}{\sin(x)} + \pi 7^x \right) dx$.

3 The Fundamental Theorem of Calculus

Theorem 3.0.0.1 (The Fundamental Theorem of Calculus)**.** Let $a, b \in \mathbb{R}, a < b$. If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Proposition 3.0.0.2 (MVT for Definite Integral)**.** Let $a, b \in \mathbb{R}, a < b$. If f cont on $[a, b]$, then

$$\exists c \in [a, b] \text{ s.t. } \int_a^b f(x)dx = f(c)(b - a).$$

Proof. Suppose f is continuous on $[a, b]$ and F is *any* entiderivative of f on $[a, b]$. Let $P = \{x_i\}_i^n$ be a Riemann Partition over $[a, b]$ where $\Delta x = \frac{b-a}{n}$. Thus,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x, \tag{*}$$

by R definition of the definite int with continuity on $[a, b]$ for any $x_i^* \in [x_{i-1}, x_i]$.

Note that by assumption, F is differentiable on $[a, b]$ and F is continuous on $[a, b]$ as differentiability $\implies$ continuity. In particular, F is continuous on each $[x_{i-1}, x_i] \subseteq [a, b]$ and F is diff on each $(x_{i-1}, x_i) \subseteq [a, b]$.
Therefore, by Mean Value Theorem (applied to F on (x_{i-1}, x_i)),

$$\begin{aligned} \exists c_i \in (x_{i-1}, x_i) \text{ s.t. } F'(c_i) &= \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} && \text{MVT} \\ \iff F'(c_i)(x_i - x_{i-1}) &= F(x_i) - F(x_{i-1}) \\ &\text{algebra; note that } x_i - x_{i-1} = \Delta x \text{ by } P \text{ and } F'(c_i) = f(c_i) \text{ by assumption} \\ \implies F(x_i) - F(x_{i-1}) &= f(c_i)\Delta x. && (**) \end{aligned}$$

We choose $x_i^* = c_i$.
Then,

$$\begin{aligned} \int_a^b f(x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x && \text{R definition of definite integral} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i^*)\Delta x && \text{by chosen } x_i^* \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) && \text{by } ** \\ &= \lim_{n\to\infty} \left((F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) + \cdots + (F(x_n) - F(x_{n-1}))\right) \\ & && \text{by definition of } \Sigma \\ &= \lim_{n\to\infty} F(x_n) - F(x_0) && \text{algebra} \\ &= \lim_{n\to\infty} F(b) - F(a) && \text{by P definition} \\ &= F(b) - F(a) && \text{by limit constant law; as required.} \end{aligned}$$

□

Example. Compute $\displaystyle\int_0^1 (x^2\sqrt{x} + e^x + \frac{1}{1+x^2})\,dx$

Solution. Let $f(x) = x^2\sqrt{x} + e^x + \frac{1}{1+x^2}$. Choose $F(x) = \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + e^x + \arctan(x)$. Note that $F'(x) = f(x)$.
Then, by fundamental Theorem of Calculus,

$$\begin{aligned}
\int_0^1 (x^2\sqrt{x} + e^x + \frac{1}{1+x^2}) &= \left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}} + e^x + \arctan(x)\right)\Bigg|_0^1 \\
&= (\frac{2}{7}(1)^{\frac{7}{2}} + e + \frac{\pi}{4}) - (0 + 1 + 0) && \text{algebra} \\
&= e + \frac{\pi}{4} - \frac{5}{7} && \text{algebra; as desired.}
\end{aligned}$$

■

4 Functions Defined by Integrals

Theorem 4.0.0.1 (FTOC II)**.** Let $a, b \in \mathbb{R}, a < b$. If f is continuous on $[a, b]$ and define $F(x) = \int_a^x f(t)dt$ for any $x \in [a, b]$, then

i. F is continuous on $[a, b]$.

ii. F is diff on (a, b).

iii. $F'(x) = f(x), \forall x \in [a, b]$, i.e., F is antiderivative of f on $[a, b]$:

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x) \text{ on } [a, b].$$

Remarks 4.0.0.1.1 (Area Accumulation)**.** $F(x) = \int_a^x f(t)dt$ from FTOC II is called an *area accumulation* function of f.

Example. Let $H(x) = \int_x^4 e^{t^2+1}dt$. Find $H'(x)$.

Solution. Note $f(t) = e^{t^2+1}$ is a composition of the exponential e^t and the polynomial t^2+1, both of which are continuous on their domain $\mathbb{R}$. Hence, f is continuous on $\mathbb{R}$.
In particular, f is continuous on, without lose of generality, $[4, x] \subset \mathbb{R}$.
Define $F(x) = \int_4^x f(t)dt$.
Thus,

$$\begin{aligned}
H'(x) &= \frac{d}{dx}\left(\int_x^4 f(t)dt\right) \\
&= \frac{d}{dx}\left(-\int_4^x f(t)dt\right) && \text{by } \textstyle\int \text{ properties} \\
&= -\frac{d}{dx}\left(\int_4^x f(t)dt\right) && \text{by diff rules} \\
&= -f(x) && \text{by FTOC II since } F'(\cdot) = f(\cdot) \\
&= -e^{x^2+1}
\end{aligned}$$

■

Proof of FTOC II. Suppose f is continuous on $[a, b]$ and define $F(x) = \int_a^x f(t)\,dt$ for any $x \in [a, b]$. WTS: F is continuous on $[a, b]$ and F is diff on (a, b) and
$F'(x) = f(x), \forall x \in [a, b]$. It suffices to show that $F'(x) = f(x), \forall x \in [a, b]$ as it necessitates differentiability on $[a, b] \supseteq (a, b)$ which further implies continuity on $[a, b]$.
Case I. (Interior Points)
Let $x \in (a, b)$ be arbitrary. Consider

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} && \text{by definition of } F' \\
&= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} && \text{by def of } F \\
&= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt && \text{demonstrated in lecture} \\
&= \lim_{h\to 0} \frac{1}{(x+h) - x} \int_x^{x+h} f(t)\,dt && \text{by algebra: ``}\tfrac{1}{b-a}\int_a^b f(t)\,dt\text{''} \\
&= \lim_{h\to 0} f(c)
\end{aligned}$$

For some $c \in [x, x+h]$ by MVT for int since f is constant on $[x, x-h] \subseteq [a, b]$ where c depends on h

$$\begin{aligned}
&(*)\text{Note } h \to 0 \iff |x + h - x| \to 0 \iff |c - x| \to 0 \\
&= \lim_{c\to x} f(c) && \text{by } (*) \\
&= f(x) && \text{since } f \text{ is continuous at } x \text{ as } x \in [a, b].
\end{aligned}$$

Case II.
Let $x = a$ and $x = b$. WTS $F'_+(a) = f(a)$ and $F'_-(b) = f(b)$. This follows by analogous argument to case I but change the 2-sided limits to appropriate LH and RH limits and replace $x = b$ and $x = a$ respectively.
Remark. The reason we consider two cases is because for the first case, any derivative in $[a, b]$ is a two-sided limit. Whereas for the end point of the interval, a and b it is one side limit. □

Example. Let $g(x) = \displaystyle\int_{\sin(x)}^{\cos(x)} \arctan(t)\,dt$. Find $g'(x)$.

Solution. Let $f(t) = \arctan(t)$. Since arctan is an inverse trig function, and so continuous on $\mathrm{dom}(f) = \mathbb{R}$. In particular, f is continuous on, w.l.o.g., $[\sin(x), \cos(x)] \subset \mathbb{R}$ as the interval symmetric. Define $F(x) = \displaystyle\int_c^x f(t)\,dt$ where c is a constant s.t. $c \in [\sin(x), \cos(x)]$. So,

$$\begin{aligned}
g'(x) &= \frac{d}{dx}\left(\int_{\sin(x)}^{\cos(x)} f(t)\,dt\right) \\
&= \frac{d}{dx}\left(\int_{\sin(x)}^{c} f(t)\,dt + \int_{c}^{\cos(x)} f(t)\,dt\right) \\
&\qquad \text{by union interval property; for any constant } c \text{ between } \sin(x) \text{ and } \cos(x) \\
&= \frac{d}{dx}\left(-\int_{c}^{\sin(x)} f(t)\,dt + \int_{c}^{\cos(x)} f(t)\,dt\right) && \text{by def int prop} \\
&= \frac{d}{dx}\left(-F(\sin(x)) + F(\cos(x))\right) && \text{by definition of } F \\
&= -F'(\sin(x))\cos(x) + F'(\cos(x)) \cdot (-\sin(x)) && \text{by diff rule} \\
&= -f(\sin(x))\cos(x) - f(\cos(x))\sin(x) && \text{by FTOC II} \\
&= -\cos(x)\arctan(\sin(x)) - \sin(x)\arctan(\cos(x)) && \text{by definition of } f\text{; as required.}
\end{aligned}$$

■

5 Integration Techniques

5.1 Integration Techniques

5 **Concerned Integrals:**

i. Chapter 4 Methods (analytical, geometrical, inspection, FTOC)

ii. Substitution Rule

iii. Integration by Parts

iv. Partial Fraction Decomposition

v. Trigonometric Substitution

Theorem 5.1.0.1 (Subsitution Rule)**.** If $f(x), g(x)$ and $f(g(x))g'(x)$ are continuous, then For definite integral,

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du \qquad \text{where } u = g(x) \text{ and } du = g'(x)dx$$

For indefinite integral,

$$\int f(g(x))g'(x)dx = \int f(u)du \qquad \text{where } u = g(x) \text{ and } du = g'(x)dx$$

Proof (of definite integral). Suppose $f(x), g(x)$ and $f(g(x))g'(x)$ are continuous on $[a, b]$. Let $u = g(x)$ and $du = g'(x)dx$. By FTOC II we know F is an antiderivative of f on $[-, \cdot]$. For the right hand side,

$$\begin{aligned}
&\int_{g(a)}^{g(b)} f(u)du \\
&= F(u)\Big|_{g(a)}^{g(b)} && \text{by FTOC I} \\
&= F(g(b)) - F(g(a))
\end{aligned}$$

Claim: $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ on $[a, b]$. Let $x \in [a, b]$ be arbitrary. Then,

$$\begin{aligned}
(F(g(x)))' &= F'(g(x)) \cdot g'(x) \\
&= f(g(x)) \cdot g'(x) && \text{as } F' = f.
\end{aligned}$$

Thus our claim holds. Then, for the left hand side,

$$\begin{aligned}
&\int_a^b f(g(x))g'(x)dx \\
&= F(g(x))\Big|_a^b && \text{by our claim that holds} \\
&= F(g(b)) - F(g(a)).
\end{aligned}$$

Note, since $F(g(b)) - F(g(a)) = F(g(b)) - F(g(a))$ it follows that

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du \qquad \text{where } u = g(x) \text{ and } du = g'(x)dx\text{; as required.}$$

□

Example. Find $\int \frac{e^{\arctan(x)}}{1+x^2} dx$.

Solution. Let $u = \arctan(x)$ and $du = \frac{1}{x^2} dx$. Then,

$$\begin{aligned}
\int \frac{e^{\arctan(x)}}{1+x^2} dx &= \int e^{\arctan(x)} \cdot \frac{1}{1+x^2} dx \\
&= \int e^u du && \text{where } u = \arctan(x) \text{ and } du = \tfrac{1}{x^2} dx \text{ as defined} \\
&= e^u + C && \text{by inspection} \\
&= e^{\arctan(x)} + C && \text{by defined } u.
\end{aligned}$$

■

Example. Find $\int_0^{\frac{\pi}{2}} \sin^5(x) \cos^3(x) dx$.

Solution. Let $u = \sin(x)$ and $du = \cos(x) dx$.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^5(x)\cos^3(x) dx &= \int_0^{\frac{\pi}{2}} \sin^5(x)\cos^2(x)\cos(x) dx \\
&= \int_0^{\frac{\pi}{2}} \sin^5(x)(1-\sin^2(x))\cos(x) dx \\
&= \int_0^{\frac{\pi}{2}} \sin^5(x)(1-\sin^2(x))\cos(x) dx \\
&= \int_0^1 u^5(1-u^2) du && \text{by chosen } u \text{ and } du \\
&= \int_0^1 u^5 - u^7 du && \text{algebra} \\
&= \left(\frac{u^6}{6} - \frac{u^8}{8}\right)\Bigg|_0^1 && \text{by FTOC I} \\
&= \frac{1}{6} - \frac{1}{8} - (0) \\
&= \frac{1}{24} && \text{as required.}
\end{aligned}$$

■

Exercise. Solve using $u = \cos(x)$.

Example. Find $\int \sqrt{2-x}dx$.

Solution. Let $u = 2 - x. du = -dx$. Then,

$$\begin{aligned}
\int \sqrt{2-x}dx &= \int \sqrt{2-x} \cdot dx. \\
&= \int u^{\frac{1}{2}} \cdot -du \\
&= -\int u^{\frac{1}{2}} du \\
&= -\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C \\
&= -\frac{2}{3}(2-x)^{\frac{3}{2}} + C && \text{by chosen } u\text{; as required.}
\end{aligned}$$

■

Exercise. Find $\int \sqrt{4-\sqrt{x}}dx$.

Example. Evaluate $\int \sqrt{3+x^2}x^5 dx$.

Solution. Choose $u = 3 + x^2$ and $du = 2xdx$.

$$\begin{aligned}
\int \sqrt{3+x^2}x^5 dx &= \int \sqrt{u}(u-3)^2 \frac{du}{2} && \text{by chosen } u \text{ and } du \\
&= \frac{1}{2}\int \sqrt{u}(u-3)^2 du \\
&= \frac{1}{2}\int u^{\frac{5}{2}} - 6u^{\frac{3}{2}} + 9u^{\frac{1}{2}} du \\
&= \frac{1}{2}\left(\frac{u^{\frac{7}{2}}}{\frac{7}{2}} - \frac{6u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{9u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C \\
&= \frac{(3+x^2)^{\frac{7}{2}}}{7} - \frac{6}{5}(3+x^2)^{\frac{5}{2}} + 3(3+x^2)^{\frac{3}{2}} + C
\end{aligned}$$

■

Theorem 5.1.0.2 (Integration by Parts)**.** If $u = f(x)$ and $v = g(x)$ are diff, then

i. For definite int $\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$.

ii. For indefinite int $\int udv = uv - \int vdu$.

Proof of ii. We show if $u = f(x)$ and $v = g(x)$ are diff then $\int udv = uv - \int vdu$, i.e.,

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

Suppose $u = f(x)$ and $v = g(x)$ are diff. We know

$$\begin{aligned} (f(x)g(x))' &= f'(x)g(x) + f(x)g'(x) && \text{by product rule} \\ \iff f(x)'g(x) &= (f(x)g(x))' - g(x)f'(x) \\ \implies \int f(x)g'(x)dx &= \int ((f(x)g(x))' - g(x)f'(x))dx && \text{Integrate wrt } x \\ &= \int ((f(x)g(x))' - \int g(x)f'(x)dx && \text{by indef int prop} \\ &= f(x)g(x) - \int g(x)f'(x)dx && \text{by definition of indef int} \end{aligned}$$

□

Theorem 5.1.0.3 (Partial Fraction Decomposition)**.**
Remarks. The propose of PFD is to rewrte certain *proper* rational functions into equivalent partial fractions.
For example, $f(x) = \frac{5x+11}{(x+3)(x+2)} = \frac{4}{x+3} + \frac{1}{x+2}$. Recall a rational function is a function of the form $\frac{P(x)}{Q(x)}$ s.t. P and Q are polynomials and $Q(x) \neq 0$. Where a polynomial is a function of the form $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ for any $a_i \in \mathbb{R}, n \in \mathbb{Z}^{\geq 0}$ By *proper* we refer to the state in which $\deg(P(x)) < \deg(Q(x))$.

Example Write the pfd form only for the followings:

i. $f(x) = \frac{8x-12}{(x+6)(x-3)}$

Solution.

$$\frac{8x-12}{(x+6)(x-3)} = \frac{A}{x+6} + \frac{B}{x-3} \qquad \text{by rule 1 where } A, B \in \mathbb{R}$$

■

ii. $f(x) = \frac{9-9x}{x^3(x^2+1)}$

Solution.

$$\frac{9-9x}{x^3(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Mx+N}{x^2+1}$$

by rule 1 and rule 2 where $A, B, C, M, N \in \mathbb{R}$

■

iii. $f(x) = \frac{7}{(x-1)^2(2x^2+7)^2(ex-\pi)}$

Solution.

$$\frac{7}{(x-1)^2(2x^2+7)^2(ex-\pi)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{2x^2+7} + \frac{Mx+N}{(2x^2+7)^2} + \frac{K}{ex-\pi}$$

by rule 1 and rule 2 where $A, B, C, D, M, N, K \in \mathbb{R}$

■

Example. Evaluate $\int \frac{x+5}{x^2+x-2}dx$.

Solution. Note

$$\begin{aligned}\frac{x+5}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1} &= \frac{A(x-1)+B(x+2)}{(x+2)(x-1)} \\ &\implies x+5 = A(x-1)+B(x+2) \\ &\implies A=-1 \wedge B=2 && \text{choose } x=1 \text{ and } x=-2\end{aligned}$$

$$\begin{aligned}\int \frac{x+5}{x^2+x-2}dx &= \int \frac{x+5}{(x+2)(x-1)}dx && \text{for } A,B\in\mathbb{R} \text{ by rule 1} \\ &= \int \frac{A}{x+2} + \frac{B}{x-1}dx && \text{by pfd rule 1 where } A,B\in\mathbb{R} \\ &= \int \frac{-1}{x+2} + \frac{2}{x-1}dx && \text{by solved } A\wedge B \\ &= -\ln|x+2| + 2\ln|x-1| + C && \text{by inspection}\end{aligned}$$

■

Example. Find $\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2} dx$.

Solution. Note

$$\begin{aligned}
\frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2} &= \frac{5x^3 - 3x^2 + 2x - 1}{x^2(x^2+1)} \\
&= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1} && \text{by rule 1 and 2 for } A, B, C, D \in \mathbb{R} \\
&= \frac{A(x(x^2+1)) + B(x^2+1) + (Cx+D)(x^2)}{x^4 + x^2} \\
&\implies 5x^3 - 3x^2 + 2x - 1 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2 \\
&= (A + C)x^3 + (B + D)x^2 + Ax + B \\
&\implies \begin{cases} 5 &= A + C \\ -3 &= B + D \\ 2 &= A \\ -1 &= B \end{cases} \\
&\implies A = 2 \wedge B = -1 \wedge C = 3 \wedge D = -2.
\end{aligned}$$

Then,

$$\begin{aligned}
\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2} dx &= \int \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1} dx && \text{by rule 1 and 2 for } A, B, C, D \in \mathbb{R} \\
&= \int \frac{2}{x} + \frac{-1}{x^2} + \frac{3x - 2}{x^2 + 1} dx && \text{by computed } A, B, C, D \\
&= \int \frac{2}{x} + \frac{-1}{x^2} + \frac{3x}{x^2 + 1} - \frac{2}{x^2 + 1} dx \\
&= 2 \ln |x| + \frac{1}{x} + \frac{3}{2} \ln(x^2 + 1) - 2 \arctan(x) + C
\end{aligned}$$

■

Excercise. Evaluate $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$.

Theorem 5.1.0.4 (Trigonometric Substitution)**.**
Remark. The purpose is to rewrite the integrated equivalently
depending on the form: $a^2 + u^2, u^2 - a$ or $a^2 - u^2$, where $a \in \mathbb{R}^+, u$ is a polynomial. There are 3 trig subs:

i.
$$a^2 + u^2 \to u = a \tan\theta, \theta \in (\frac{-\pi}{2}, \frac{\pi}{2}).$$

ii.
$$a^2 - u^2 \to u = a \sin\theta, \theta \in [\frac{-\pi}{2}, \frac{\pi}{2}], |u| \leq a.$$

iii.
$$u^2 - a^2 \to u = a \sec\theta, \theta \in \begin{cases} [0, \frac{\pi}{2}), & \text{if } u \geq a \\ (\frac{\pi}{2}, \pi], & \text{if } u \leq -a. \end{cases}$$

Example. Can we apply a trig subst on the following? If so, what subs?

(a) $\displaystyle\int_0^\pi \frac{3x+1}{\sqrt{x^2+9}} dx$

Solution. Observe that type (i) trig sub is applicable as for $x^2 + 9, u = x \wedge a = 3$ satisfies $u^2 + a^2$. Let $x = 3\tan\theta, \theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$ ■

(b) $\displaystyle\int \frac{(x+3)(4x^2-16)^{5/3}}{\sqrt{x}} dx,\ x \geq 2.$

Solution. Observe that type (iii) trig sub is applicable as for $4x^2 - 16, u = 2x \wedge a = 4$ satisfies $u^2 - a^2$. Let $2x = 4\sec\theta, \theta \in [0, \frac{\pi}{2})$ as $x \geq 2 \iff 2x \geq 4 = a$. ■

Example. Find $\int \frac{\sqrt{x^2-9}}{x^3}dx, x \geq 3.$

Solution. Let $u = x$ and $3 = a$. We have a type (iii) trig sub. Let $x = 3\sec\theta, \theta \in [0, \frac{\pi}{2})$ as $u = x \geq 3 = a$. Thus, $dx = 3\sec\theta\tan\theta d\theta$.
Note $x^2 - 9 = (3\sec\theta)^2 - 9 = 3^2(\sec^2\theta - 1) = 3^2\tan^2\theta$. Thus,

$$\begin{aligned}
\int \frac{\sqrt{x^2-9}}{x^3}dx &= \int \frac{\sqrt{3^2\tan^2\theta}}{3^3\sec^3\theta}3\sec\theta\tan\theta\, d\theta \\
&= \int \frac{\sqrt{(3\tan\theta)^2}}{3^2\sec^2\theta}\, d\theta \\
&= \int \frac{|3\tan\theta|}{3^2\sec^2\theta}\, d\theta \\
&= \int \frac{3\tan\theta}{3^2\sec^2\theta}\, d\theta && \text{as } \tan\theta \geq 0 \text{ for } \theta \in [0, \tfrac{\pi}{2}] \\
&= \frac{1}{3}\int \frac{\tan\theta}{\sec^2\theta}\, d\theta \\
&= \frac{1}{3}\int \sin^2\theta\, d\theta \\
&= \frac{1}{3}\int \left(\frac{1-\cos(2\theta)}{2}\right) d\theta \\
&= \frac{1}{6}\int (1-\cos(2\theta))d\theta \\
&= \frac{1}{6}\left(\theta - \frac{\sin(2\theta)}{2}\right) + C \\
&\text{recall that } x = 3\sec\theta \iff \frac{x}{3} = \sec\theta \iff \theta = \sec^{-1}(\frac{x}{3}) \text{ as the interval of } \theta \text{ is injective.} \\
&= \frac{1}{6}\left(\theta - \frac{2\sin\theta\cos\theta}{2}\right) + C \\
&= \frac{1}{6}\left(\sec^{-1}(\frac{x}{3}) + \frac{\sqrt{x^2-9}}{x}\cdot\frac{3}{x}\right) + C && \text{by triangle method.}
\end{aligned}$$

■

5.2 Integration Techniques Summery

1. Geometrically: Only for def int of the forms

$$\int_c^d ax + bdx \vee \int_c^d \pm\sqrt{(r^2 - (x-h)^2}dx.$$

2. By Inspection: literally attention is all you need.

3. u-subs (inverse chain rule)

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

4. by parts (inverse product rule)

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx = uv - \int vdu.$$

5. PFD

$$\int \frac{P(x)}{Q(x)} \text{ s.t. } \deg(p) < \deg(q).$$

6. trig subs

$u^2 - a^2$	$u^2 + a^2$	$a^2 - u^2$
$u = a\sec(\theta)$	$u = a\tan(\theta)$	$u = a\sin(\theta)$
$u \geq a \implies \theta \in [0, \pi/2)$	$\theta \in (-\pi/2, \pi/2)$	$\theta \in [-\pi/2, \pi/2], \lvert u\rvert \leq a$
$u \leq -a \implies \theta \in [\pi/2, \pi]$		
$du = a\sec(\theta)\tan(\theta)$	$du = a\sec^2(\theta)$	$du = a\cos(\theta).$

5.3 Improper Integrals

In past, we have assumed, for $\int_a^b f(x)dx$,

i. $[a,b]$ is bounded.

ii. $f(x)$ is bounded on $[a,b]$, i.e., no V.A. on $[a,b]$.

If these conditions fail, we get an *improper integral.*
Example.

1. $\int_0^\infty (\arctan(x))^2 dx$ - improper due to ∞ bound (type I).
2. $\int_2^8 \frac{1}{\sqrt{x-2}} dx$ - improper due to VA at 2 (type II).
3. $\int_{-1}^1 x^{-2} dx$ - improper due to VA at 0 (type II).
4. $\int_{\pi/2}^{\pi} \csc(x) dx$ - improper due to VA at π (type II).
5. $\int_{-\infty}^1 \frac{\cos^2(2x)}{x^2+1} dx$ - improper due to $-\infty$ bound (type I).

Example. Evaluate $\int_1^\infty \frac{1}{(3x+1)^2} dx$.

Solution. Intuitively, think of A as approaching to infinity. Then,

$$\begin{aligned}
\int_1^\infty \frac{1}{(3x+1)^2} dx &= \lim_{A\to\infty} \int_1^A \frac{1}{(3x+1)^2} dx && \text{by definition of type I} \\
&= \lim_{A\to\infty} -\frac{1}{3}(3x+1)^{-}1\Big|_1^A && \text{by inspection} \\
&= \lim_{A\to\infty} -\frac{1}{3}\left(\frac{1}{3A+1} - \frac{1}{4}\right) && \text{by FTOC I} \\
&= \frac{-1}{3}\left(0 - \frac{1}{4}\right) = \frac{1}{12} && \text{by limit type } \lim_{k\to\infty} \tfrac{1}{k} = 0.
\end{aligned}$$

$\therefore$ limit exists. The integral *converges to* 1/12

■

Example. Does $\displaystyle\int_0^5 \frac{\ln(x)}{x}dx$ converge/diverge?

Solution.

$$\begin{aligned}
\int_0^5 \frac{\ln(x)}{x}dx &= \lim_{A\to 0^+}\int_A^5 \frac{\ln(x)}{x}dx && \text{by definition of type II (as } A \to 0 \text{ from the right)}\\
&= \lim_{A\to 0^+} \frac{(\ln(x))^2}{2} A^5 && \text{by inspection}\\
&= \lim_{A\to 0^+} (\frac{(\ln(5))^2}{2} - \frac{(\ln(A))^2}{2})\\
&\quad \text{by FTOC II; note that } A \to 0^- \implies \ln(A) \to -\infty \implies (\ln(A))^2 \to \infty\\
&= -\infty\\
&\therefore \text{ limit DNE. } \therefore \text{ The improper integral diverges.}
\end{aligned}$$

■

Example. Does $\int_{-1}^{1} x^{-2}dx$ converge/diverge?

Solution.

$$\begin{aligned}\int_{-1}^{1} x^{-2}dx &= \int_{-1}^{0} x^{-2}dx + \int_{0}^{1} x^{-2}dx \\ &= \lim_{A\to 0^-}\int_{-1}^{A} x^{-2}dx + \lim_{A\to 0^+}\int_{B}^{1} x^{-2}dx \qquad \text{by definition of type II.}\end{aligned}$$

Considr

$$\begin{aligned}\int_{-1}^{0} x^{-2}dx &= \lim_{A\to 0^-}\int_{-1}^{A} x^{-2}dx \\ &= \lim_{A\to 0^-} -x^{-1}\Big|_{-}1^{A} \\ &= \lim_{A\to 0^-}\left(\frac{-1}{A} + \frac{1}{-1}\right) \\ &= \infty\end{aligned}$$

Therefore, $\int_{-1}^{0} x^{-2}dx$ diverges. Now it suffices to show that $\lim_{A\to 0^+}\int_{B}^{1} x^{-2}dx$ does not evaluate to $-\infty$. Note $\forall x \in (0,-1], x^{-2} > 0$. Thus,

$$\int_{0}^{1} x^{-2}dx \geq 0.$$

In particular,

$$\int_{0}^{1} x^{-2}dx \neq -\infty.$$

Thus,

$$\int_{-1}^{1} x^{-2}dx \text{ diverges.}$$

■

Exercise. (Hint: apply union property interval.) Does $\int_{-\infty}^{1} \frac{1}{x-1}dx$ converge or diverge?

Motivation. Con/Div?

$$\int_1^\infty \frac{(\cos(2x^4+1)^{200}+1)}{(x^2+e\pi)^{100}}dx.$$

Theorem 5.3.0.1 (Comparison (Direct) Theorem)**.** Suppose f, g, h are continuous on an interval I and $\int_I f(x)dx$ is an improper int.

1. If $0 \leq f(x) \leq g(x), \forall x \in I \wedge \int_I g(x)dx$ converges, then

$$\int_I f(x)dx \text{ also converges.}$$

2. If $0 \leq h(x) \leq f(x), \forall x \in I \wedge \int_I h(x)dx$ diverges, then

$$\int_I f(x)dx \text{ also diverges.}$$

Example. Let f, g, h be continuous on interval I. Consider the improper integral $\int_I f(x)dx$.
Prove if $0 \leq f(x) \leq g(x), \forall x \in I$ and $\int_I g(x)dx$ converges, then

$$\int_I f(x)dx \text{ converges.}$$

Proof. WLOG $I = [a, \infty)$ for any $a \in \mathbb{R}$. Assume (1) $0 \leq f(x) \leq g(x), \forall x \in [a, \infty)$ and (2), $\int_a^\infty g(x)dx$.
WTS $\int_a^\infty f(x)dx$ convergences by definition, i.e., $\lim_{A\to\infty} \int_a^A f(x)dx$ exists.
Let $A \in [a, \infty)$ be arbitrary. Note,

$$\begin{aligned}
(1) &\implies 0 \leq f(x) \leq g(x), \forall x \in [a, A] \subseteq [a, \infty) \\
&\implies \int_a^A 0dx \leq \int_a^A f(x)dx \leq \int_a^A g(x)dx && \text{by def int prop} \\
&\implies 0 \leq \int_a^A f(x)dx \leq \int_a^A g(x)dx \\
&\implies 0 \leq \int_a^A f(x)dx \leq \int_a^A g(x)dx, \forall A \geq a && \text{as } A \text{ is arbitrary, i.e., } A \in [a, \infty) \\
&\implies \lim_{A\to\infty} 0 \leq \lim_{A\to\infty} \int_a^A f(x)dx \leq \lim_{A\to\infty} \int_a^A g(x)dx \\
&\implies \lim_{A\to\infty} 0 \leq \lim_{A\to\infty} \int_a^A f(x)dx \leq \int_a^\infty g(x)dx && \text{by def type I} \\
&\implies \lim_{A\to\infty} 0 \leq \lim_{A\to\infty} \int_a^A f(x)dx \leq \text{some constant } k \\
& && \text{by assumption (2) } \textstyle\int_a^\infty g(x)dx \text{ converges}
\end{aligned}$$

we know that $\lim_{A\to\infty} \int_a^A f(x)dx$ is continuous on $[a, A]$ by FTOC II
and increasing on $[0, A]$ as $f(x) \geq 0$.

$\therefore \lim_{A\to\infty} \int_a^A f(x)dx$ exists

i.e., $\int_a^\infty f(x)dx$converges.

□

6 Sequences

Definition 6.0.1 (Sequence)**.** A *sequence* is a function $a_n : N \to \mathbb{R}$ where $N \subseteq \mathbb{N}$. Typically, $|N| = \infty$. We denote an infinite sequence of real numbers in general terms as

$$\{a_n\}_{n=1}^{\infty}.$$

Definition 6.0.2 (Sequence Convergence/Divergence)**.** We say a_n converges to l iff

$$\exists l \in \mathbb{R}, \forall \varepsilon > 0, \exists N > 0 \text{ s.t. } \forall n \in \mathbb{N}, n > N \implies |a_n - l| < \varepsilon.$$

That is,

$$\lim_{n\to\infty} a_n = l.$$

We can also say a_n diverges to $\pm\infty$, i.e.,

$$\forall M > 0, \exists N > 0 \text{ s.t. } \forall n \in \mathbb{N}, n > N \implies a_n > M \vee a_n < -M.$$

If $a_n > N$, then a_n diverges to ∞ otherwise $-\infty$.

Remark. $n \in \mathbb{N} \implies n \geq 1 > 0. N \geq a \implies n > a.$

Example. Prove $a_n = \frac{n^2-2}{n^2+2n+2}$ converges to 1.

Proof. Choose $l = 1 \in \mathbb{R}$. Let $\varepsilon > 0$ be arbitrary. Choose $N = \frac{2}{\varepsilon} > 0$. Suppose $n > N$. Consider,

$$\begin{aligned}
|a_n - 1| &= |\frac{n^2-2}{n^2+2n+2} - 1| \\
&= |\frac{n^2-2-(n^2+2n+2)}{n^2+2n+2}| \\
&= |\frac{2n-4}{n^2+2n+1}| \\
&= \frac{|2||n-2|}{|n^2+2n+1|} \\
&= \frac{2(n+2)}{n^2+2n+2} && \text{assume } N > 2 \implies n > N > 2 \implies n-2 > 0 \\
&\leq \frac{2(n+2)}{n^2+2n} \\
&= \frac{2}{n} \\
&< \frac{2}{N} \\
&= \varepsilon && \text{as required.}
\end{aligned}$$

□

Example. Prove if $\{a_n\}$ and $\{b_n\}$ converge, then

$$\{a_nb_n\} \text{ converges.}$$

Proof. Suppose

1. $\{a_n\}$ conv to some $a \in \mathbb{R}$.
2. $\{b_n\}$ conv to some $b \in \mathbb{R}$.

WTS: $\{a_nb_n\}$ conv, i.e.,

$$\exists l \in \mathbb{R}, \forall \varepsilon > 0, \exists N > 0 \text{ s.t. } \forall n \in \mathbb{N}, n > N \implies |a_nb_n - l| < \varepsilon.$$

Choose $l = ab \in \mathbb{R}$.
Let $\varepsilon > 0$ be arbitrary.
Note 1. $\implies \exists N_1 > 0$ s.t. $n > N_1 \implies |a_n - a| < \frac{1}{|b|+1}\frac{\varepsilon}{2}$.
Note 2. $\implies \exists N_2 > 0$ s.t. $n > N_2 \implies |b_n - b| < \frac{1}{|a|+1}\frac{\varepsilon}{2}$.
Choose $N = \max\{N_1, N_2, N_3\} > 0$.
Note that 2. $\implies \exists N_3 > 0$ s.t. $n > N_3 \implies |b_n - b| < 1$. From which it follows that $|b_n| = |b_n - b + b| \leq |b_n - b| + |b| < 1 + |b|.(*)$
Suppose $n > N$. Then,

$$\begin{aligned}
|a_nb_n - ab| &= |a_nb_n + 0 - ab| \\
&= |a_nb_n + ab_n - ab_n - ab| \\
&= |b_n(a_n - a) + a(b_n - b)| \\
&\leq |b_n(a_n - a)| + |a(b_n - b)| && \text{trig inequality} \\
&= |b_n||a_n - a| + |a||b_n - b| && \text{def of } |\cdot| \\
&< (1 + |b|)|a_n - a| + |a||b_n - b| && \text{by } (*) \\
&< (1 + |b|)\frac{1}{1 + |b|}\frac{\varepsilon}{2} + \frac{|a|}{1 + |a|}\frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + (1)\frac{\varepsilon}{2} && \text{as } \tfrac{|a|}{1+|a|} < \tfrac{|a|+1}{1+|a|} = 1. \\
&= \varepsilon.
\end{aligned}$$

□

Theorem 6.0.2.1 (Convergent Sequence Properties)**.** Let $\{a_n\}$ and $\{b_n\}$ be sequences. Let $a, b \in \mathbb{R}$. If $a_n \to a$ and $b_n \to b$, then

1. $\{a_n + b_n\}$ converges to $a + b$

2. any $c \in \mathbb{R}$, $\{ca_n\}$ converges to ca

3. $\{a_n b_n\}$ converges to ab

4. $\{\frac{a_n}{b_n}\}$ converges to $\frac{a}{b}$ where $b \neq 0$ and $b_n \neq 0$.

Theorem 6.0.2.2. If $\{a_n\}$ conv, then $\{a_n\}$'s limit is unique.

Proof. Assume $\{a_n\}$ conv to both (1) $l_1 \in \mathbb{R}$ and (2) $l_2 \in \mathbb{R}$. WTS $l_1 = l_2 \iff l_1 - l_2 = 0$. It suffices to prove

$$\forall \varepsilon > 0, |l_1 - l_2| < 0.$$

Remark. The sufficiency can be demonstrated by its incompatibility with $l_1 \neq l_2 \iff l_1 < l_2 \vee l_1 > l_2$. Suppose either case, then we can show a contradiction, therefore it is the equivalent of $l_1 = l_2$.

Let $\varepsilon > 0$ be arbitrary.

Note

$$\begin{cases} (1)\, \exists N_1 > 0 \text{ s.t. } n > N_1 \text{ then } |a_n - l_1| < \frac{\varepsilon}{2} \\ (2)\, \exists N_2 > 0 \text{ s.t. } n > N_2 \text{ then } |a_n - l_2| < \frac{\varepsilon}{2} \end{cases}$$

Consider

$$\begin{aligned} |l_1 - l_n| &= |l_1 + 0 - l_2| \\ &= |l_1 - a_n + a_n - l_2| \\ &= |-(a_n - l_1) + (a_n - l_2)| \\ &\leq |-(a_n - l_1)| + |a_n - l_2| && \text{trig inq} \\ &= |(a_n - l_1)| + |a_n - l_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{provided } n > \max\{N_1, N_2\} \\ &= \varepsilon && \text{as needed.} \end{aligned}$$

□

Definition 6.0.3 (Boundedness and Monotonicity)**.** Let $\{a_n\}$ be a sequence. We say

1. $\{a_n\}$ is bounded if it is bdd above and below, i.e.,

$$\exists c \in \mathbb{R}^{\geq 0} \text{ s.t. } |a_n| \leq c, \forall n \in \mathbb{N}.$$

2. $\{a_n\}$ is monotone if $\{a_n\} \uparrow \forall i \in \mathbb{N} \vee \{a_n\} \downarrow \forall i \in \mathbb{N}$.
   Example. $\{1 + (-1)^n\}$ is bounded and not monotone.
   Example. e^n is not bounded but monotone.

Theorem 6.0.3.1 (Bounded Monotone Convergent Theorem–BMCT)**.** If $\{a_n\}$ is bounded and monotone, then

$$\{a_n\} \text{ converges.}$$

Proof. Suppose $\{a_n\}$ is strictly increasing and bounded above. WTS $\{a_n\}$ converges by definition. WTS $\exists l \in \mathbb{R}, \forall \varepsilon > 0, \exists N > 0$ s.t. $n > N \implies |a_n - l| < \varepsilon$. Consider $A = \{a_n | n \in \mathbb{N}\} \subset \mathbb{R}$.We know $A \neq \emptyset$ as $a_1 \in A$. Then, A is bounded above by assumption. By completeness axiom, $\sup(A)$ exists. Let such $\sup(A) = \alpha$. Choose $l = \alpha \in \mathbb{R}$. Consider arbitrary $\varepsilon > 0$. Choose $N \in \mathbb{N}$ s.t. $\alpha - \varepsilon < a_N$. Suppose $n > N$. Then, by assumption of strictly increasing and chosen N by approximation theorem,

$$\begin{aligned} \alpha - \varepsilon < a_N < a_n \leq \alpha < \alpha + \varepsilon \\ \implies \alpha - \varepsilon < a_n < \alpha + \varepsilon \\ \implies |a_n - \alpha| < \varepsilon \end{aligned} \qquad \text{as required.}$$

x □

Example. Let $\{a_n\}$ be sequence defined by
$a_1 = \sqrt{6}, a_{n+1} = \sqrt{6 + a_n}$ if $n \in \mathbb{N}$ and $n \geq 1$.
(Recursively–self-referenced-defined sequence). Prove $\{a_n\}$ converges.
Rough, note

$$\begin{aligned}
a_1 &= \sqrt{6} \\
a_2 &= \sqrt{6 + a_1} = \sqrt{6 + \sqrt{6}} \\
a_3 &= \sqrt{6 + a_2} = \sqrt{6 + \sqrt{6 + \sqrt{6}}} \\
\vdots & \quad\ \vdots \quad\quad\quad \vdots
\end{aligned}$$

In particular, as $0 < 6 < 9$,

$$\begin{aligned}
\sqrt{6} &< \sqrt{9} = 3 \\
\sqrt{6 + \sqrt{6}} &< \sqrt{6 + \sqrt{9}} = \sqrt{9} = 3 \\
\sqrt{6 + \sqrt{6 + \sqrt{6}}} &< \sqrt{6 + \sqrt{6 + \sqrt{9}}} = \sqrt{6 + \sqrt{9}} = \sqrt{9} = 3. \\
& \quad\quad\quad\quad\quad\quad\quad\quad \vdots \quad\quad\quad \vdots
\end{aligned}$$

Claim: $\{a_n\}$ is (1) bounded above and (2) strictly increasing.

Proof. of (1) WTS $\exists M \in \mathbb{R}, \forall n \in \mathbb{N}, a_n \leq M$. Choose $M = 3 \in \mathbb{R}$.
WTS $a_n < 3, \forall n \in \mathbb{N}$. Consider $n = 1$ for the base case. Then, $a_1 = \sqrt{6} < \sqrt{9} = 3$. Thus, the base case holds.
Inductive Step: $\forall k \in \mathbb{N}, (a_k < 3 \implies a_{k+1} < 3)$. Let $k \in \mathbb{N}$ be arbitrary.
Assume $a_k < 3$ (Induction Hypothesis). WTS: $a_{k+1} < 3$. Consider a_{k+1}, by definition

$$\begin{aligned} a_{k+1} &= \sqrt{6 + a_k} \\ &< \sqrt{6 + 3} && \text{as } 6 + a_k < 6 + 3 \text{ by I.H.} \\ &= 3. \end{aligned}$$

By PMI, $a_n < 3, \forall n \in \mathbb{N}$ i.e., $\{a_n\}$ is strictly bounded above by 3.
of (2) WTS: $\forall n \in \mathbb{N}, a_n < a_{n+1}$. Let $n \in \mathbb{N}$ be arbitrary. Consider

$$\begin{aligned} a_n^2 - a_{n+1}^2 &= a_n^2 - (\sqrt{6 + a_n})^2 && \text{by definition of } \{a_n\} \\ &= a_n^2 - (6 + a_n) \\ &= a_n^2 - a_n - 6 \\ &= (a_n - 3)(a_n + 2) \\ &< 0 && \text{as } a_n \in (0, 3) \text{ by (1) thus the left hand evaluates to negative} \\ &\iff a_n^2 < a_{n+1}^2 \\ &\implies a_n < a_{n+1} && \text{since } \sqrt{\ } \text{ is increasing as required.} \end{aligned}$$

Therefore by BMCT, $\{a_n\}$ converges.

□

7 Series

Definition 7.0.1 (Series)**.** Let $\{a_n\}$ be an infinite sequence. Then, $a_1 + \cdots + a_n$ is an infinite series. In particular,

$$S_n = a_1 + \cdots + a_n = \sum_{n=1}^{\infty} a_n.$$

Definition 7.0.2 (Convergent Series)**.** Given $\sum a_n$, we say $\sum a_n$ converges if

$$\{S_n\} \text{ converges.}$$

That is,

$$\exists S \in \mathbb{R} \text{ s.t. } \lim_{n\to\infty} S_n = S.$$

If $\sum a_n$ does not converges, we say $\sum a_n$ diverges.

Example. $\sum_{n=1}^{\infty} \ln(\frac{n+1}{n})$ converges or not.

Proof. We first observe that $\ln(\frac{n+1}{n}) = \ln(n+1) - \ln(n)$. Consider

$$\begin{aligned} S_n &= a_1 + \cdots + a_n \\ &= (\ln(2) - \ln(1)) + (\ln(3) - \ln(2)) + \cdots + (\ln(n) - \ln(n-1)) + (\ln(n+1) - \ln(n)) \\ &\qquad \text{note that it telescopes} \\ &= \ln(n+1) - \ln(1) \\ &= \ln(n+1). \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty} \ln(n+1) \\ &= \infty \\ &\therefore \lim \text{ DNE }. \end{aligned}$$

Thus the series diverges. ∎

Theorem 7.0.2.1 (Properties of Convergent Series)**.** If $\sum a_n$ and $\sum b_n$ both converges, then

i. $\sum(a_n + b_n)$ converges.

ii. $\forall c \in \mathbb{R}, \sum c(a_n)$ converges.

iii. $\lim\limits_{n\to\infty} a_n = 0$ (vanishing condition).

Proof of iii. Suppose $\sum a_n$ converges, i.e., $\lim\limits_{n\to\infty} S_n = s$ for some $s \in \mathbb{R}$. Consider

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} a_n + 0 \\
&= \lim_{n\to\infty} (a_1 + \cdots + a_{n-1}) + a_n - (a_1 + \cdots + a_{n-1}) \\
&= \lim_{n\to\infty} S_n - S_{n-1} \\
&= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \\
&\text{by assumption the series converges therefore individual limit exists} \\
&= s - s \\
&= 0 && \text{as required.}
\end{aligned}$$

□

Theorem 7.0.2.2 (Divergence Test)**.** Given $\sum a_n$. If $\lim\limits_{n\to\infty} a_n \neq 0$, then

$$\sum a_n \text{ diverges.}$$

Proof. Recall $P \implies Q \equiv \neg Q \implies \neg P$. Note the contrapositive of the theorem is our vanishing condition which has been shown already. □

Example. Does $\sum_{n=1}^{\infty} \frac{\sqrt{5n^4+1}}{n^2+3}$ conv or div?

Proof. Note that

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{\sqrt{5n^4+1}}{n^2+3} \\ &= \lim_{n\to\infty} \frac{\sqrt{5n^4+1}}{n^2+3} \cdot \frac{1/n^2}{1/n^2} \\ &= \lim_{n\to\infty} \frac{\sqrt{1/n^4}\sqrt{5n^4+1}}{1+3/n^2} \\ &= \lim_{n\to\infty} \frac{\sqrt{5+1/n^4}}{1+3/n^2} \\ &= \sqrt{5}.\end{aligned}$$

By divergence test theorem, this suffices for us to conclude that $\sum_{n=1}^{\infty} \frac{\sqrt{5n^4+1}}{n^2+3}$ diverges. □

Definition 7.0.3 (Geometric Series)**.** Let $a, r \in \mathbb{R}$ s.t. $a \neq 0$. A series of the form

$$a + ar + ar^3 + \cdots + ar^n + \cdots = \sum_{n=0}^{\infty} ar^n$$

is a *geometric series.* The number r is the ratio of the G.S.

Example.

1. $1 - e + e^2 - e^3 + \ldots$ – yes $r = -e$.
2. $\sum_{n=3}^{\infty} \pi(\frac{1}{2})^n$ – yes $r = \frac{1}{2}$.
3. $\sum_{n=1}^{\infty} \frac{n}{e^n}$ – counter-example, note that the ratio changes.

Example. Consider $\sum_{n=0}^{\infty} ar^n, a, r \in \mathbb{R}, a \neq 0$. For what r values does this G.S. con or div?

Proof. We prove by def, i.e. ,$\lim_{n\to\infty} S_n =???$. We know $S_n = a + ar + ar^2 + \cdots + ar^{n-1} + ar^n$ by definition. From which it follows that

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n + ar^{n+1}.$$

Thus,

$$\begin{aligned} S_n - rS_n &= (a + ar + ar^2 + \cdots + ar^{n-1} + ar^n) - (ar + ar^2 + ar^3 + \cdots + ar^n + ar^{n+1}) \\ &= 1 - ar^{n+1} \\ &\implies S_n(1-r) = a(1 - r^{n+1}) \\ &\implies S_n = \frac{a(1-r^{n+1})}{1-r} && \text{for } r \neq 1; \\ &\implies S_n = a + a + \cdots + a(1)^n = (n+1)a && \text{for } r = 1 \end{aligned}$$

Case I., suppose $r = 1$.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (n+1)a = \pm\infty \qquad \text{for } a > 0 \vee a < 0.$$

Thus the limit does not exists. Therefore the geometric series diverge by definition in this case. Case II., suppose $r \neq 1$.

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r} && \text{for } a > 0 \vee a < 0. \\ &= \frac{a}{1-r} \lim_{n\to\infty} 1 - r^{n+1} \\ &= \frac{a}{1-r}(1 - \lim_{n\to\infty} r^{n+1}) \qquad = \frac{a}{1-r} && \text{for } |r| < 1 \\ &\text{else,} && \text{if } |r| > 1 \vee r = -1. \end{aligned}$$

Since

$$\{r^n\} = \begin{cases} 0, & |r| < 1 \\ \infty, & r > 1 \\ \text{DNE}, & r < -1 \\ \text{DNE}, & r = -1. \end{cases}$$

Thus, by definition, $\sum ar^n$ converges to some $\frac{a}{1-r}$ if $|r| < 1$
and diverges if $|r| > 1 \vee r = \pm 1$. □

Theorem 7.0.3.1 (GS Test)**.** Given $\sum_{n=0}^{\infty} ar^n, a, r \in \mathbb{R}, a \neq 0$.

1. If $|r| < 1$, then the series converges with
$$\sum ar^n = \frac{a}{1-r}.$$
2. If $|r| \geq 1$ then our series diverges.

Example.

a. $\sum_{n=3}^{\infty} n!$

Proof. Note $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n! = \infty \neq 0$. Thus, by div test it diverges. □

b. $\sum_{n=2}^{\infty} \pi(\frac{7}{13})^n$

Proof. By GS test, as $|r| = \frac{7}{13} < 1$, the series converges to $\frac{a}{1-r}$, i.e., $\frac{\pi(7/13)^2}{1-\frac{7}{13}}$. □

c. $\sum_{n=0}^{\infty} \left((\frac{1}{2})^n - e(\frac{4}{7})^n\right)$

Proof. Consider $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ By GS, $|r| = \frac{1}{2} < 1$. Thus, it converges to $\frac{1}{1-\frac{1}{2}} = 2$. Consider $\sum_{n=0}^{\infty} e\left((\frac{4}{7})\right)^n$. By GS, as $|r| = \frac{4}{7} < 1$, it converges to $\frac{e}{1-\frac{4}{7}} = \frac{7e}{3}$. By properties of convergent series, $\sum_{n=0}^{\infty} \left((\frac{1}{2})^n - e(\frac{4}{7})^n\right)$ converges as 'conv − conv = conv'. In particular, the given sum converges to $2 - \frac{7e}{3}$. □

Theorem 7.0.3.2 (Integral Test)**.** Given $\sum a_n$. If $f(x)$ is positive, continuous, decreasing on $[1, \infty)$ and $a_n = f(n), \forall n \in \mathbb{N}$ then,

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)dx \text{ converges.}$$

Thus, by logical equivalence,

$$\sum_{n=1}^{\infty} a_n \text{ diverges} \iff \int_1^{\infty} f(x)dx \text{ diverges}$$

Proof. □

Example. Consider $\sum_{n=3}^{\infty} ne^{-n}$. Conv? Div?

Proof. Let $a_n = ne^{-n} = f(n), \forall n \in \mathbb{N}$ s.t. $n \geq 3$. Then, for $x \in [3, \infty), f(x) = xe^{-x}$. Note that $f > 0$ as product of positives is positive.
For $x \in [3, \infty), f'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x)$ since $x \in [3, \infty), f'(x) < 0 \forall x \in [3, \infty)$, thus decreasing on the interval of interest. As differentiability implies continuity, f is continuous. Thus the hypothesis of integral test is satisfied. Consider

$$\begin{aligned}
\int_3^{\infty} xe^{-x}dx &= \lim_{A\to\infty} \int_3^{A} xe^{-x}dx && \text{type I} \\
&= \lim_{A\to\infty} \left([-xe^{-x}]_3^A + [-e^{-x}]_3^A \right) \\
&= \lim_{A\to\infty} \left(\frac{-A}{e^A} + \frac{3}{e^3} - \frac{1}{e^A} + \frac{1}{e^3} \right) \\
&= \frac{4}{e^3}
\end{aligned}$$

□

Definition 7.0.4 (p series)**.** Let $p \in \mathbb{R}^+$. A series of the form

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

is a *p-series.* The number p is called the $p - value$.

Example.

1. $\sum_{n=1}^{\infty} \frac{1}{n}$ - $p = 1$.
2. $\sum_{n=2}^{\infty} \frac{1}{n^{7 \cdot 8+\pi}}$ - yes $p = 7 \cdot 8 + \pi$.
3. $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots$ - yes $p = \frac{1}{2}$.
4. $\sum_{n=1}^{\infty} \frac{1}{n^n}$ - no, as p varies.

Theorem 7.0.4.1 (p-sereis test)**.** Given $\sum_{n=1}^{\infty} \frac{1}{n^p}, p \in \mathbb{R}^+$.

1. If $p \leq 1$, then
$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ diverges.}$$
2. If $p > 1$, then
$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges.}$$

Proof. Let $a_n = \frac{1}{n^p} := f(n), \forall n \in \mathbb{N}$. Thus, $f(x) = \frac{1}{x^p}, \forall x \in [1, \infty)$. For, $x \in [1, \infty), f(x) = \frac{1}{x^p} > 0$. Consider two cases for f'. Suppose $p \neq 1$, then $f'(x) = (x^{-p})' = -px^{-(p+1)} < 0$. Consider $\int_1^{\infty} f(x)dx = \int_1^{\infty} x^{-p}dx$. [left as exercise by integral test] $\square$

Example. $\sum_{n=1}^{\infty} \frac{1}{n}$. Conv, or Div. Note that $\frac{1}{n}$ is a p series with $p = 1$. Note $p = 1 \leq 1$. Thus, by $p-$series test, $\sum_{n=1}^{\infty} \frac{1}{n}$.

Theorem 7.0.4.2 (Direct Comparison Theorem for Series)**.** Given $\sum a_n, \sum b_n, \sum c_n$.

1. If $0 \leq a_n \leq b_n, \forall n \in \mathbb{N}$ and $\sum b_n$ conv, then
$$\sum_{n=1}^{\infty} a_n \text{ also converges.}$$

2. If $0 \leq c_n \leq a_n, \forall n \in \mathbb{N}$ and $\sum c_n$ div, then
$$\sum_{n=1}^{\infty} a_n \text{ also diverges.}$$

Proof of 1. Suppose $0 \leq a_n \leq b_n, \forall n \in \mathbb{N}$ and $\sum b_n$ conv.
WTS: $\sum a_n$ converges, i.e., $\lim_{n\to\infty} = S_n$ exists, i.e., $\{S_n\} = \{S_1, \ldots, S_n\}$ converges.
Recall $S_n = a_1 + \cdots + a_n$. Let $n \in \mathbb{N}$ be arbitrary. Consider

$$\begin{aligned} S_{n+1} &= S_n + a_{n+1} && \text{by definition; note that } S_n \geq 0 \land a_{n+1} \geq 0 \text{ by assumption} \\ &\geq S_n && \text{minimize sum of non-negative terms} \end{aligned}$$

Hence, $\forall n \in \mathbb{N}, S_{n+1} \geq S_n, i.e.,$ $\{S_n\}$ is increasing. By assumption,

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &\leq \sum_{n=1}^{\infty} b_n && \text{as } a_n \leq b_n, \forall n \in \mathbb{N} \\ &= t && \text{such that for some } t \in \mathbb{R} \text{ by assumption} \end{aligned}$$

Note that by max of sum and $a_n \geq 0, \forall n \in \mathbb{N}$.

$$S_n \leq \sum_{n=1}^{\infty} a_n.$$

By transitivity, it follows that $S_n \leq t, \forall n \in \mathbb{N}$. Thus,

$$\exists t \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N} S_n \leq t$$

that is, $\{S_n\}$ is bounded above by t.
$\therefore$ by BMCT, $\{S_n\}$ converges, i.e., $\lim_{n\to\infty} S_n$ exists, i.e.,

$$\sum a_n \text{converges by definition.}$$

$\square$

Example. $\sum_{n=3}^{\infty} \frac{\arctan(n)}{n^{\frac{1}{2}}+4^n}$.

Proof. I., For $n \in \mathbb{N}$ s.t. $n \geq 3, a_n = \frac{\arctan(n)}{n^{1/2}+4^n} > 0$ are all positive. II., find a good comparison. Consider

$$\begin{aligned}
\frac{\arctan(n)}{n^{\frac{1}{2}}+4^n} &< \frac{\arctan(n)}{4^n} && \text{minimize denominator} \\
&\leq \frac{\pi/2}{4^n} && \text{properties of arctan} \\
&= \frac{\pi}{2} \cdot \frac{1}{4^n} && \text{as } \pi < 4
\end{aligned}$$

note that this is a geometric series with $r = \frac{1}{4} < 1, a = \frac{\pi}{2}$

Consider $\sum_{n=3}^{\infty} b_n = \sum_{n=3}^{\infty} \frac{\pi}{2}(\frac{1}{4})^n$. As $0 \leq a_n \leq b_n, \forall n \in \mathbb{N}$ and $\sum b_n$ converges, by Direction Comparsion,

$$\sum_{n=1}^{\infty} a_n \text{ also converges.}$$

□

Definition 7.0.5 (Alternating Series)**.** A series of the form

$$b_1 \pm b_2 \mp b_3 \pm \cdots = \sum_{i=1}^{\infty} (-1)^{n+1} b_n, \text{ s.t. } b_n > 0$$

is called an *alternating series.*

Example.

1. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ is AS.

2. $\sum_{0=1}^{\infty} \frac{\cos(n\pi)}{n!}$ is AS in disguise.

Theorem 7.0.5.1 (Alternating Series Test - AST)**.** Given $\sum_{i=1}^{\infty}(-1)^{n+1}b_n, b_n > 0$.
If (1), $b_n \geq b_{n+1}, \forall n \in \mathbb{N}$ and (2), $\lim_{n\to\infty} b_n = 0$, then

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \text{ converges.}$$

Proof. Given $\sum_{i=1}^{\infty}(-1)^{n+1}b_n, b_n > 0$.
Suppose $b_n \geq b_{n+1}, \forall n \in \mathbb{N}$ and (2), $\lim_{n\to\infty} b_n = 0$. WTS: $\sum_{i=1}^{\infty}(-1)^{n+1}b_n$ converges. □

Example. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ conv? div?

Proof. This is an AS with $b_n = \frac{1}{\ln(n)} > 0$. Consider $\lim_{n\to\infty} = \frac{1}{\ln(n)} = 0$.
Note that for arbitrary $n \geq 2 \in \mathbb{N}$

$$n < n+1 \implies \ln(n) < \ln(n+1) \implies \frac{1}{\ln(n)} \geq \frac{1}{\ln(n+1)} \implies b_n \geq b_{n+1}, \forall n \geq 2 \in \mathbb{N}.$$

As both conditions for AST suffice, the given series converges. WTS $\lim_{n\to\infty} S_n$ exists, i.e., $\{S_n\}$ converges. 'Prove using BMCD $\{2_n - 1\}$ conv and $\{2_n\}$ conv and $l_1 = l_2$.' ☐

Definition 7.0.6 (Absolute/Conditional convergent)**.** A series $\sum a_n$ is said to:

1. *Absolutely Converge* iff
$$\sum |a_n| \text{ conv.}$$
2. *Conditionally Converge* iff
$$\sum |a_n| \text{ div and } \sum a_n \text{ conv.}$$

Example. Does $\sum_{n=1}^{\infty} \frac{\sin(6n)}{4^n}$ CC, AC, div?

Proof. Consider

$$\begin{aligned}
\sum_{n=1}^{\infty} \left|\frac{\sin(6n)}{4^n}\right| &= \sum_{n=1}^{\infty} \frac{|\sin(6n)|}{4^n} && \text{properties of } |\cdot| \\
&\leq \sum_{n=1}^{\infty} \frac{1}{4^n} && \text{as sin is bounded above by 1} \\
&= \frac{1/4}{3/4} = \frac{1}{3} && \text{geometric series}
\end{aligned}$$

As a series above is convergent, by CT, $\sum_{n=1}^{\infty} |\frac{\sin(6n)}{4^n}|$ converges, thus the given series absolutely converges. □

Example. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n}$. Consider $\lim_{n\to\infty} \frac{1}{n} = 0$. $\forall n \in \mathbb{N}$,

$$n < n+1 \implies \frac{1}{n} > \frac{1}{n+1} \implies b_n > b_{n+1}.$$

By AST,

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} \text{converges.}$$

Consider $\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{n}$ by $|\cdot|$ def. Note that as this is a p-series with $p = 1$, it diverges. Thus the given series conditionally converges.

Theorem 7.0.6.1 (Ratio Test)**.** Given $\sum a_n$ series such that $a_n \neq 0, \forall n \in \mathbb{N}$. Define

$$L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| \text{ s.t. } L \in [0, \infty) \cup \{\infty\}.$$

1. If $L < 1$, then
$$\sum a_n \text{ AC thus also conv.}$$
2. If $L > 1$, then
$$\sum a_n \text{ div.}$$
3. If $L = 1$, then
$$\text{inconclusive.}$$

Proof. Given $\sum a_n, a_n \neq 0, \forall n \in \mathbb{N}$. Define

$$L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| \text{ s.t. } L \in [0, \infty) \cup \{\infty\}.$$

I.e.,

$$\begin{cases} L \geq 0, & \forall \varepsilon > 0, \exists N > 0 \text{ s.t. } \forall n \in \mathbb{N}, n > M \implies |\frac{a_{n+1}}{a_n}| < \varepsilon. \\ L = \infty, & \forall M > 0, \exists N > 0 \text{ s.t. } \forall n \in \mathbb{N}, n > N \implies |\frac{a_{n+1}}{a_n}| > M. \end{cases}$$

Outline.

1. If $L < 1$, we use bounding of PMI to show $|a_n|$ is $\leq$ a GS: use CT.
2. if $L > 1$, we use bounding + div test to show $\sum a_n$ div.

□

Example. Does $\sum_{n=0}^{\infty} \frac{e^n}{(2n)!}$ conv? div?

Proof. Find $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \left(\frac{\frac{e^{n+1}}{(2(n+1))!}}{\frac{e^n}{(2n)!}} \right) = \lim_{n\to\infty} \frac{e^{n+1}}{e^n} \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} \frac{e}{(2n+2)(2n+1)}$.
Note that

$$\lim_{n\to\infty} \frac{e}{(2n+2)(2n+1)} = 0.$$

Thus, by RT, the given series converges. □

8 Power Series

Definition 8.0.1 (Power Series)**.** Let $\{c_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$. A series of the form

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is a *power series centered at a*. Where a is the center of a series and c_n is the n^{th} *term coefficient of power series.*

Example.

1. $1 + x + x^2 + \cdots + x^n = \sum_{n=0}^{\infty} x^n$ is a power series with $a = 0 \wedge c_n = 1$.
2. $\sum_{n=1}^{\infty} \frac{(x-3)^n}{4^n}$ is a PS with $a = 3 \wedge c_n = \frac{1}{4^n}$.
3. $\sum_{n=2}^{\infty} \frac{(4x+1)^n}{\sqrt{n}2^n}$ is a PS with $a = -1/4$ and $c_n = \frac{2^n}{\sqrt{n}}$

Definition 8.0.2 (Taylor Series)**.** Let f be a function that has derivatives of all orders at $x = a$. A power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n \text{ where } c_n = \frac{f^{(n)}(a)}{n!}$$

is called a *Taylor Series* for f If $a = 0$, the Taylor Series is called a *MacLaurin Series.*

Example. Let $f(x) = e^x$. Find a Maclaurin Series for this function.

Solution. $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ by definition.

$$\begin{aligned} f^{(0)}(x) = f(x) = e^x &\implies f(0) = 1 \\ f'(x) = e^x &\implies f'(0) = 1 \\ f''(x) = e^x &\implies f''(0) = 1. \end{aligned}$$

Claim $f^{(n)}(x) = e^x \implies f^{(n)}(0) = 1$. Thus the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{1 \cdot x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, \forall x \in \mathbb{R}.$$

■

Definition 8.0.3 (Radius of Convergences)**.** Let $a \in \mathbb{R}$. Given $\sum c_n(x-a)^n$. The largest value $R \in \mathbb{R}^+ \cup \{\infty\}$ s.t. PS converges absolutely for x satisfying$|x - a| <$ and diverges for x satisfying $|x - a| > R$, is called the *radius of convergence* of the power series.
The *interval of convergences* of PS, $I = \{x \in \mathbb{R} : \sum c_n(x-a)^n \text{ converges}\}$.

Example. $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$. Find I. First note that this is a power series with $a = -2 \wedge c_n = n/3^{n+1}$.

Solution. Find/compute R. (ratio test). Consider

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| &= \lim_{n\to\infty} \left| \frac{(\frac{(n+1)(x+2)^{n+1}}{3^{n+2}})}{\frac{n(x+2)^n}{3^{n+1}}} \right| \qquad \text{where } x \neq a \\
&= \lim_{n\to\infty} \frac{n+1}{n} \frac{3^{n+1}}{3^{n+2}} \left| \frac{(x+2)^{n+1}}{(x+2)^n} \right| \\
&= \lim_{n\to\infty} (1 + \frac{1}{n}) \frac{1}{3} |x+2| \\
&= \frac{1}{3} |x+2| \lim_{n\to\infty} (1 + \frac{1}{n}) \\
&= \frac{|x+2|}{3}
\end{aligned}$$

By RT, our series AC if $\frac{|x+2|}{3} < 1$ and div if $\frac{|x+2|}{3} > 1$.
Thus, PS will AC for $|x+2| < 3$ and div for $|x+2| > 3$. Thus, $R = 3$.
Then, we check end points for $x = a \pm R$. Consider $x = -5$., then we have $\sum_{n=0}^{\infty} \frac{n(-5+2)^n}{3^{n+1}}$ We apply div test. Suppose n is even then, $a_n = \infty$ else to $-\infty$. Thus, the series diverges. Thus, our PS must div at $x = 5$. Consider $x = 1$. Consider $\sum_{n=0}^{\infty} \frac{n(3^n)}{3^n \cdot 3} = \sum_{n=0}^{\infty} \frac{n}{3}(1)^n$. Div test the series diverges as $\lim_{n\to\infty} \frac{n}{3} = \infty$. Thus, we conclude that our PS diverges at $x = 1$. Thus,

$$I = (-5, 1).$$

■