Cantors' Diagnol Argument (1891)

References to Professor Philip Kremer's Lecture in PHLC51.

Theorem 0.0.0.1 (Reals Strictly Dominate Naturals)**.**
For the set of naturals, $\mathbb{N}$ and the set of reals, $\mathbb{R}$,

$$\mathbb{N} \prec \mathbb{R}.$$

Proof. We show that (1) $\mathbb{N} \preceq \mathbb{R}$ and (2) $\mathbb{N} \not\approx \mathbb{R}$.
First, as $\mathbb{N} \subseteq \mathbb{R}$, it follows that $\mathbb{N} \preceq \mathbb{R}$. Thus, we have shown (1) as required.
Now, suppose to the contrary $\mathbb{N} \approx \mathbb{R}$. Thus, there is a one-one function f from $\mathbb{N}$ onto $\mathbb{R}$. Consider such function f with equinumerous assignment:

$$\begin{array}{ccc} f(0) & = & 0.\underline{1}314 \quad \dots \\ f(1) & = & 0.2\underline{6}79 \quad \dots \\ f(2) & = & 0.64\underline{2}5 \quad \dots \\ f(3) & = & 0.474\underline{3} \quad \dots \\ \vdots & \vdots & \vdots \ \vdots \ \vdots \ \vdots \quad \ddots \end{array}$$

until all real numbers are mapped.
Now let d_n be $(n+1)^{th}$ diagonal, i.e., the $(n+1)^{th}$ digit after decimal of $f(n)$ underlined.
We define a Cantor number $c = 0.c_0c_1c_2 \dots c_n$ s.t.

$$c_n = \begin{cases} 9, & \text{if } d_n \in \{0,1,2,3,4\} \\ 1, & \text{if } d_n \in \{5,6,7,8,9\} \end{cases}$$

where c_n is the $(n+1)^{th}$ digit of c.
Then, $c_n \neq d_n, \forall n \in \mathbb{N}$ by construction. $(*)$
However, as f is from $\mathbb{N}$ onto $\mathbb{R}$, every outputs, including c must be mapped by some $m \in \mathbb{N}$. Consider such m for $f(m) = c$. Then,

$$f(m) = c = 0.c_0c_1c_2 \dots c_m$$

where c_m is the $(m+1)^{th}$ digit of c.
But the $(m+1)^{th}$ diagonal also, by definition, happen to be d_m.
From which it follows that $c_m = d_m$. As n is arbitrary, by $(*)$, we have derive a contradiction where

$$c_m \neq d_m \land c_m = d_m.$$

Thus, it must be that $\mathbb{N} \not\approx \mathbb{R}$. Therefore it is demonstrated that

$$\mathbb{N} \prec \mathbb{R}, \text{i.e., Reals strictly dominate Naturals.}$$

□