Application

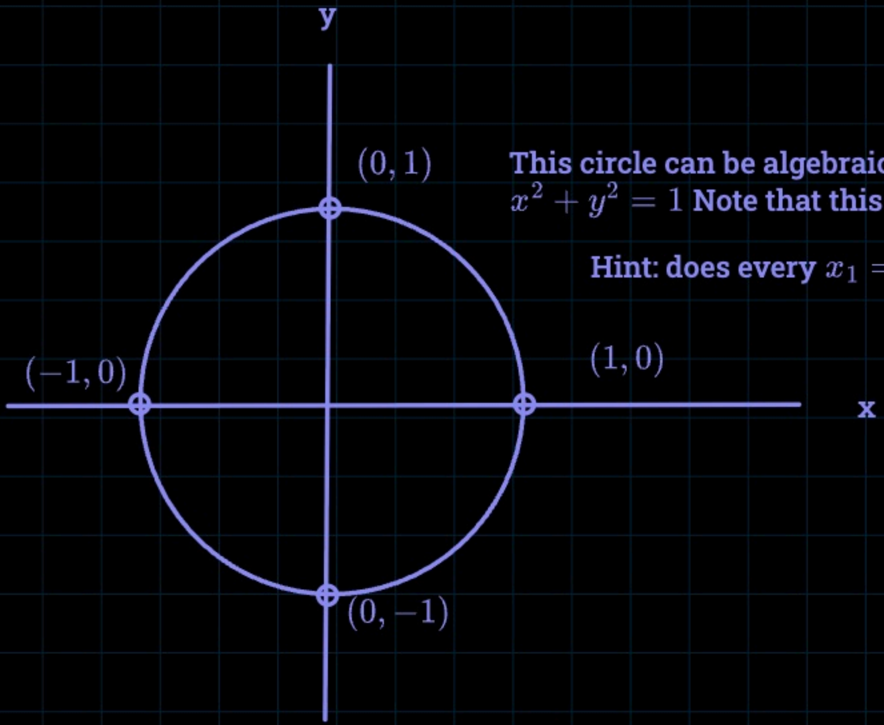
Consider

1. Let $f(x) = x^2 + 2$, compute $f(2)$.
2. Let $f(x) = y = x + 4$, find the average rate of change for f on $x \in [1, 4]$. I.e., find $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(4) - f(1)}{4 - 1}$
3. Consider $x_1 = 6$. Suppose new x_2 is 12. What is the relative change of x? Hint (use $\Delta x / x_1 = \frac{x_2 - x_1}{x + 1}$.)
4. $f(x) = x^3 + 3$. What is the derivative of f with respect to x? i.e., find $\frac{df}{dx}$. Hint: Use Power Rule!!
5. Consider $h(x) = \frac{x+3}{x+2}$ what is the derivative of h with respect to x? Hint: Use Quotient Rule.
6. Consider $g(x) = (x + 7)^4$ what is the derivative of g with respect to x? Hint: Use the Chain Rule.

Understandings:

1. Discuss the difference between Δx and dx. Think about this question: why would $\frac{dy}{dx} = \frac{\Delta y}{\Delta x}$ on a line, say $y = x$??

2.



This circle can be algebraically described by
$x^2 + y^2 = 1$ Note that this (IS/IS NOT) a function. Why?

Hint: does every $x_1 = x_2$ produces one and only one y?

In particular, a branch of mathematics, called Calculus is often deployed.

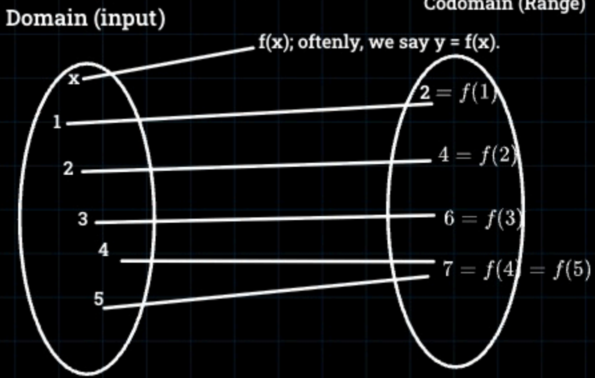
$y = x + 1$ ←代數(algebra) we can also call this a function.

The question then arises. What exactly is a function: f. We define function such that given an input, it produces an unique output.

Remark. What does unique mean?
Unique means when we have two things having the same properties, they must be equal

i.e.,
$x_1 = x_2 \implies f(x_1) = f(x_2)$

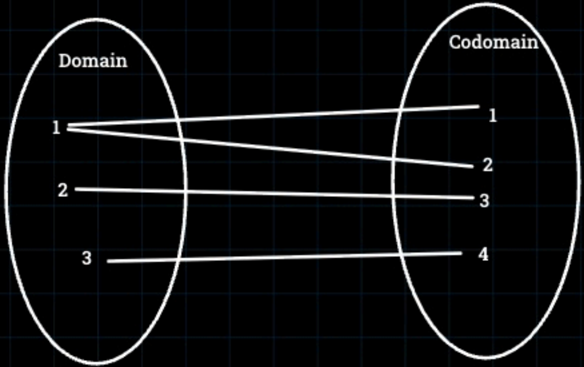
Example. A function



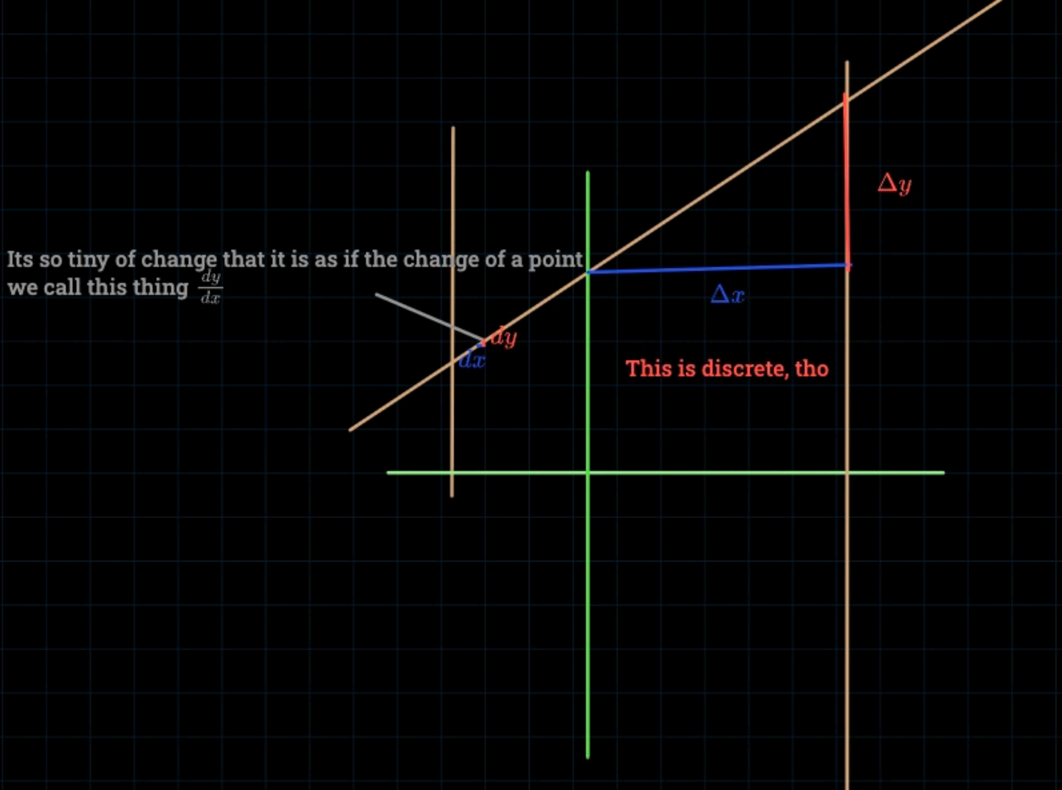
f is a function, and when we say $f(1) = 2$ this means when we input 1 to f, f out put 2.

Example. Find a function that satisfy $f(1) = 2$.
Solution.
Consider
$f := x + 1, \forall x$. Then, $f(1) = 1 + 1 = 2$ such function satisfy the required.

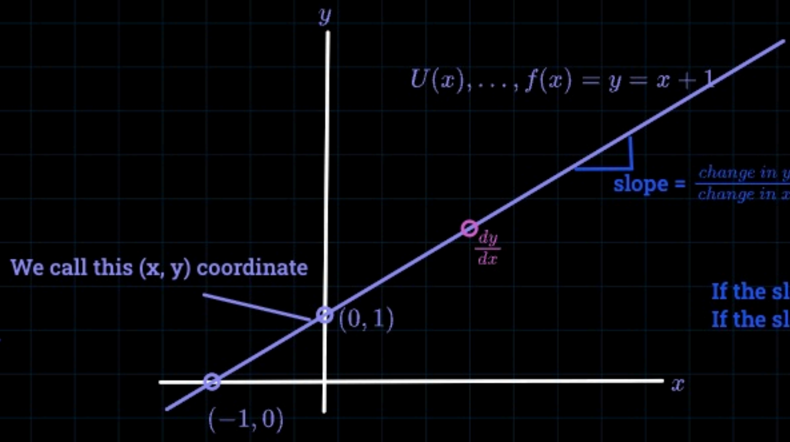
Example. (Not a function). Consider the following mappings.



Consider $x_1 = x_2 = 1$. Note that however, $f(x_1) = f(1) = 1 \neq 2 = f(1) = f(x_2)$


Its so tiny of change that it is as if the change of a point
we call this thing $\frac{dy}{dx}$
dy
dx
Δy
Δx
This is discrete, tho

We want to know something more about a function. So say, geometrically,



slope = $\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{dy}{dx}$ in this case it is 1 = first derivative.

If the slope > 0, then the function, here, $y = x + 1$ is said to be increasing;
If the slope < 0, then the function is said to be decreasing.

In Calculus, we are interested in the infinitesimal change, usually denoted as dx.

Example. (Calculate slope).

Suppose we have a function, $y = x + 1$. We would like to know the value for $\frac{\Delta y}{\Delta x}$

Say say we consider two points: 10 and 15.
A simple calculation tells us $y(10) = 11; y(15) = 16$.
So we have $\Delta y = y(15) - y(10) = 16 - 11 = 5$. Also note that $\Delta x = 15 - 10 = 5$.
Thus, $\frac{\Delta y}{\Delta x} = \frac{5}{5} = 1$. So the slope of $y = x + 1$ is 1.

Definition. Rate of change: the rate of change of a function at a point (or a segment) is defined to be the change in output (y) per change in (x), i.e., when we treat y as a function output of x that is $f(x) = y$.

Important Note:
HERE, the rate of change defined is not tangent. It is secent, i.e., the average rather than the insantaneous.

Let's now investigate the difference between discrete change and continuous change: i.e.,

$\Delta x \ v.s. \ dx$. On the left hand, Δx can be computed by the difference of the interested segment, i.e., $x_2 - x_1$. Whereas, on the right hand, dx 'cannot' be computed. In fact, $dx \approx 0$. So, the notion rate of change of a point, $\frac{dy}{dx}$ i.e., the change of y per super small change in x.

What is a theorem?

A theorem alike a theory attempts to generalize and explain a particular phenomenon. In particular, a theorem is a proven theory.

Now, we provide a theorem for the notion of instantaneous rate of change.
(Here we skip the proof.)--I will send the proof to you if you are interested!
It involves something called limit!

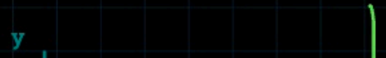
Let f, g be differentiable everywhere functions, then the followings hold:

1. Power Rule: $f(x) = ax^n + c \, such \, that \, a, c \in \mathbb{R}; \frac{d}{dx} f(x) = \frac{df}{dx} = anx^{n-1}$.
Example: $f(x) = y = 7000000000000x^{20} + 10$. Then, $f'(x) = \frac{df}{dx} = \frac{dy}{dx} = 7000000000000 \times 20 \times x^{20-1} + 0 = 7000000000000 \times 20 \times x^{19}$.
2. Chain Rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \times g'(x)$ where $'$ means derivative, i.e., $\frac{d}{dx}$.
Example: $f(x) = x^2; g(x) = x + 1$. Then, $f(g(x)) = f(x+1) = (x+1)^2$. Thus, $\frac{d}{dx} f(g(x)) = \frac{d}{dx}(x+1)^2 = 2(x+1) \times \frac{d}{dx}[x+1] = 2(x+1) \times 1$

3. Quotient Rule: $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$. Example, $f(x) = x + 1; g(x) = x + 2$; then $\frac{f(x)}{g(x)} = \frac{x+1}{x+2}$. By the theorem, we have

$\frac{d}{dx} \frac{x+1}{x+2} = \frac{(x+1)'(x+2) - (x+1)(x+2)'}{(x+2)^2} = \frac{(x+2) - (x+1)}{(x+2)^2}$.

Note for the example in 2. and 3., power rule (1.) is used to find for example, f'(x)!!!



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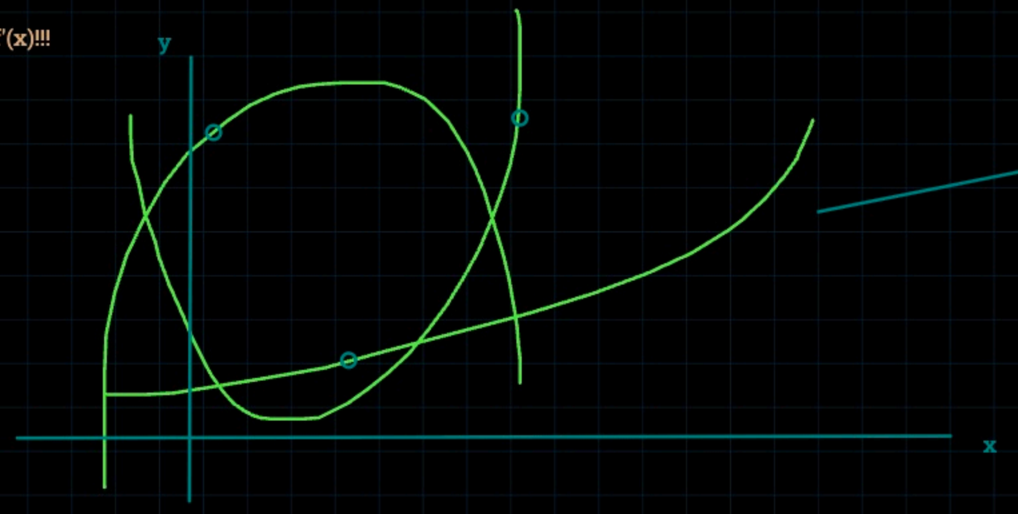
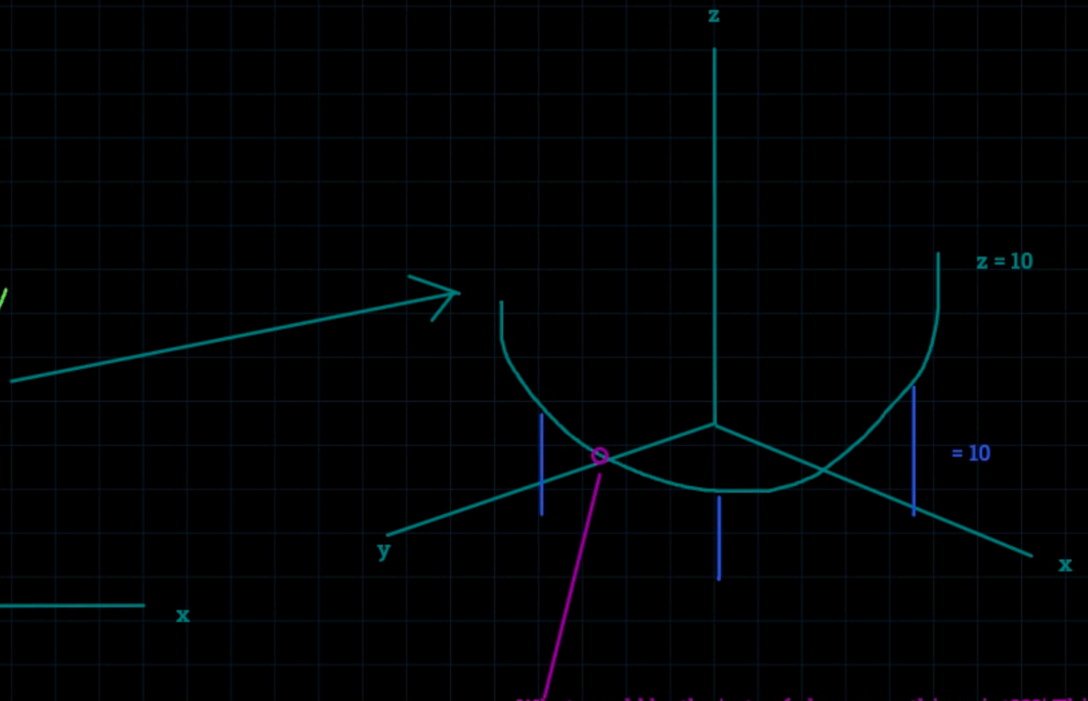
Side Note: we sometimes use partial derivative, denoted as
$\frac{\partial}{\partial x}$ which measures the change in output treating all variables constant



we would also like to know about the change of these function on a point



What would be the 'rate of change on this point???' This is when we use calculus!